Solution of EPD by Fourier Transform Method

Joy Mutegi[a], Iyaya Wanjala[b], Isaac Chepkwony[c]
[a]Mathematics Department, Multimedia University of Kenya, P.O. Box 15653-00503, Nairobi, Kenya
[b]Mathematics Department, Multimedia University of Kenya, P.O. Box 15653-00503, Nairobi, Kenya
[c]Mathematics Department, Kenyatta University of Kenya, P.O. Box 43844-001, Nairobi, Kenya

Abstract**—**We solve the Euler-Poisson-Darboux (EPD) equation using the Fourier transform method. The inverse Fourier transform is found using a convolution with the Heaviside step function in order to obtain the solution. We then extend our results into a generalized hypergeometric form and we also discuss the differentiability of the solution.

Keywords-*Euler-Poisson-Darboux (EPD) equation, Fourier transform, Heaviside step function, hypergeometric function.*

*****

The Euler-Poisson-Darboux (EPD) equation is a second order differential equation which after some transformation may be denoted by

$$\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t}-\sum_{i=1}^{n}\left(\frac{\partial^2 u}{\partial x_i^{\,2}}+\frac{\lambda_i}{x_i^2}u\right)=L(u)=0 \quad (1.1)$$

$$u(x_i,0)=f(x);\; u_t(x_i,0)=0,\; i=1,2,... \quad (1.2)$$

where k and λ are real or complex parameters and t is the time variable. The equation (1.1) called singular if the operator

coefficient $\frac{k}{t}\to\infty$ as $t\to 0$.

It is called degenerate if $t\frac{\partial^2 u}{\partial t^2}\to 0$ as $t\to 0$.

If $k=0$, and $\lambda_i=0$ (1.1) reduces to the wave equation.

The EPD equation has been studied since the ancient time of Euler (1770). Poisson (1823) solved the equation for $n=1$. An expository of the theory of Euler and Poisson was given by Darboux(1915).

Solutions for (1.1) and (1.2) have also been found for various values of k as follows;

For $k=n-1$, the solution as given by Asgeirsson (1937),

For $k>n-1$; the solution was given by Weinstein (1952)

for $k<n-1$, but $k\neq -1.-3,-5,....$ by Weinstein (1954),

for $k=-1,-3,-5,$ the solution was found by Blum (1954).

Most recently Manyonge et al (2013), Seilkhanova,(2015), Iyaya and Chepkwony (2016), Iyaya et.al (2018) among others.

The EPD appears in many fields of mathematics and physics including propagation of sound waves (Copson, 1975), theory of surfaces (Darboux, 1972), colliding gravitational fields (Hauser and Ernest, 1989), gas dynamics, etc.

We now define some important terms which are anticipated in our further discussions;

Definition 1: Hypergeometric series

The hypergeometric series ${}_2F_1(a,b;c;z)$ which is a solution of the hypergeometric differential equation is defined as

$$\begin{aligned}{}_2F_1(a,b;c;z)&=1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+...\\&=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\\&=\sum_{n=0}^{\infty}A_n z^n\end{aligned} \quad (1.3)$$

where

2- refers to the number of parameters in the numerator

1-refers to the number of parameters in the denominator

$A_n=\frac{(a)_n(b)_n}{(c)_n n!}$; and

$|z|<1,\; c\neq 0,-1,-2,...$

The Pochhammer symbols are defined by

$$(a)_0=1,\;(a)_1=a,\;(a)_2=a(a+1),,...etc$$

Generally,

$$(a)_n=a(a+1)(a+2)...(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)} \text{ and}$$

$$(1)_n=n!$$

Definition 2: Bessel function

The Bessel function of order ν is defined by

$$J_\nu(z)=\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\nu+1;-\frac{z^2}{4}\right) \qquad (1.4)$$

Definition 3 : Integral representation of the hypergeometric function

The integral representation of the hypergeometric function due to Pochhammaer is

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt \qquad (1.5)$$

The integral representation of the hypergeometric function with no parameter on the numerator and one parameter on the denominator is

$${}_0F_1(;b;z)=\frac{2\Gamma(b)}{\sqrt{\pi}\,\Gamma\left(b-\frac{1}{2}\right)}\int_0^1\left(1-t^2\right)^{b-\frac{3}{2}}\cosh\left(2\sqrt{zt}\right)dt;$$

$$\operatorname{Re}(b)>\frac{1}{2} \qquad (1.5)'$$

Definition 4: Convergence of hypergeometric function

Consider the series

$$F(a,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}$$

By use of the standard ratio test we obtain

$$\lim_{n\to\infty}\left|\frac{(a)_{n+1}(c)_n}{(a)_n(c)_{n+1}}\frac{z^{n+1}}{z^n}\frac{n!}{(n+1)}\right|$$

$$=\left\{\lim_{n\to\infty}\left|\frac{(a+n)}{(c+n)(n+1)}\right|\right\}|z|\to 0 \qquad (1.6)$$

The series (1.6) called the Confluent hypergeometric function converges to zero as $n\to\infty$.

This series is connected to the hypergeometric series and is obtained as a limit of $F\left(a,b;c;\frac{z}{b}\right)$ as $b\to\infty$.

$$F(a,c;z)=\lim_{b\to\infty}F\left(a,b;c;\frac{z}{b}\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{\left(\frac{z}{b}\right)}{n!} \qquad (1.7)$$

Hence

$$\lim_{b\to\infty}\left\{\lim_{n\to\infty}\left|\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}}\frac{(c)_n}{(a)_n(b)_n}\frac{\left(\frac{z}{b}\right)^{n+1}}{\left(\frac{z}{b}\right)^n}\frac{n!}{(n+1)}\right|\right\}$$

$$=\lim_{b\to\infty}\left\{\lim_{n\to\infty}\left|\frac{(a+n)(b+n)}{(c+n)(n+1)}\right|\right\}\left|\frac{z}{b}\right|\Bigg\}\to 0 \qquad (1.8)$$

Definition 5: Differentiation of the hypergeometric function

The derivative of the function $F(a,b;c;z)$ is given as

$$\frac{d}{dx}F(a,b;c;z)=\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}=\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^{n-1}}{(n-1)!} \qquad (1.9)$$

We let $m=n-1\Rightarrow n=m+1;\ n=1,\ m=0$.

Hence

$$\frac{d}{dz}F(a,b;c;z)=\sum_{m=0}^{\infty}\frac{(a)_{m+1}(b)_{m+1}}{(c)_{m+1}}\frac{z^m}{m!}$$

$$=\frac{ab}{c}\sum_{m=0}^{\infty}\frac{(a+1)_m(b+1)_m}{(c+1)_m}\frac{z^m}{m!}$$

$$=\frac{ab}{c}F(a+1,b+1;c+1;z) \qquad (1.10)$$

The general formula based on repeated differentiation is

$$\frac{d^k}{dz^k}F(a,b;c;z)=\sum_{m=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^m}{m!}=\frac{(a)_k(b)_k}{(c)_k}F(a+k,b+k;c+k;z) \qquad (1.11)$$

Similarly,

$$\frac{d}{dz}F(;c;z)=\sum_{m=0}^{\infty}\frac{1}{(c)_{m+1}}\frac{z^m}{m!}=\frac{1}{c}\sum_{m=0}^{\infty}\frac{1}{(c+1)_m}\frac{z^m}{m!}=\frac{1}{c}F(c+1;z) \qquad (1.12)$$

and general formula based on repeated differentiation is

$$\frac{d^k}{dz^k}F(;c;z)=\sum_{m=0}^{\infty}\frac{1}{(c)_k}\frac{z^m}{m!}=\frac{1}{(c)_k}F(c+k;z) \qquad (1.13)$$

ISSN: 2454-4248
13 – 19

2. SOLUTION OF EPD BY FOURIER TRANSFORM METHOD

We solve the Euler-Poisson-Darboux (EPD) equation using the Fourier transform method. The inverse Fourier transform of the solution will be found using a convolution with the Heaviside step function in order to obtain the solution. We then extend our results into a generalized hypergeometric form and we also discuss the differentiability of the solution.

We will solve the singular Cauchy problem in the space of distributions with respect to the space coordinate. Let $t \geq 0$ and $\underline{x} \in \mathbb{R}^n$

Consider the EPD (1.1) Subject to the initial conditions (1.2). For $\lambda_i = 0$ we take the Fourier transform wrt x on the RHS of (1.1) to get

$$\int \frac{\partial^2 u}{\partial x_i^2} e^{-i\varsigma x} dx = \sum_{i=1}^{n} |\varsigma_i^2| \bar{u} = A(\varsigma)\bar{u}. \tag{2.1}$$

where $|\varsigma| = \sqrt{\varsigma_1^2 + \varsigma_1^2 + \ldots \varsigma_n^2}$,

$$|\varsigma|^2 = \varsigma_1^2 + \varsigma_2^2 + \ldots \varsigma_n^2$$

$$= \sum_{i=1}^{n} |\varsigma_i^2| = A(\varsigma)$$

We again take the Fourier transform of the LHS of (1.1) wrt x and combine it with (2.1) to get

$$\frac{\partial^2 \bar{u}}{\partial t^2} + \frac{k}{t}\frac{\partial \bar{u}}{\partial t} + A(\varsigma)\bar{u} = 0 \tag{2.2}$$

Which belongs to the family of Bessel equation due to the presence of $\frac{k}{t}$.

Now taking Fourier transform on the initial conditions (1.2) we get

$$\bar{u}(\varsigma, 0) = \bar{f}(\varsigma) \tag{2.3}$$

and

$$\bar{u}_t(\varsigma, 0) = 0 \tag{2.4}$$

Depending on the value of k, we can choose a suitable transformation to solve (1.1).

Let $k = n-1$ and referring to Manyonge et al (2013), the EPD has been solved using the transformation relation

$$\bar{u} = t^{-(\frac{n-2}{2})} V(\theta) \;\; ; \;\; \theta = |\varsigma| t = \sqrt{A} t \tag{2.5}$$

we get

$$\frac{\partial u}{\partial t} = -\frac{(n-2)}{2} t^{-\frac{n}{2}} v + t^{-\frac{(n-1)}{2}} \frac{\partial v}{\partial \theta}\sqrt{A} \tag{2.6}$$

$$\frac{\partial^2 u}{\partial t^2} =$$

$$\frac{(n-2)}{2}\left(\frac{n}{2}\right) t^{-\frac{(n+2)}{2}} v - \frac{(n-2)}{2} t^{-\frac{n}{2}} \frac{\partial v}{\partial \theta}\sqrt{A}$$

$$-\frac{(n-2)}{2} t^{-\frac{n}{2}} \frac{\partial v}{\partial \theta}\sqrt{A} + t^{--\frac{(n-2)}{2}} \frac{\partial^2 v}{\partial \theta^2} A$$

$$= t^{-\frac{(n-2)}{2}} \left[\frac{(n-2)}{2}\left(\frac{n}{2}\right) t^{-2} v - \frac{(n-2)}{t} \frac{\partial v}{\partial \theta}\sqrt{A} + \frac{\partial^2 v}{\partial \theta^2} A \right] \tag{2.7}$$

Substituting (2.4), (2.5) and (2.6) in (1.1),

$$^{-\frac{(n-2)}{2}} \left[\frac{(n-2)}{2}\left(\frac{n}{2}\right) t^{-2} v - \frac{(n-2)}{t} \frac{\partial v}{\partial \theta}\sqrt{A} + \frac{\partial^2 v}{\partial \theta^2} A \right]$$

$$+ \frac{n-1}{t}\left[-\frac{(n-2)}{2} t^{-\frac{n}{2}} v + t^{-\frac{(n-2)}{2}} \frac{\partial v}{\partial \theta}\sqrt{A} \right] + At^{-\frac{(n-2)}{2}} v = 0$$

or

$$A\frac{\partial^2 v}{\partial \theta^2} + \frac{\sqrt{A}}{\theta}\left(\sqrt{A}\right)\frac{\partial v}{\partial \theta} + \frac{4\theta^2 - (n-2)^2}{4\theta^2} Av = 0$$

or

$$\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{\theta}\frac{\partial v}{\partial \theta} + \left[1 - \left(\frac{\frac{n-2}{2}}{\theta} \right) \right] v = 0 \tag{2.8}$$

which is a Bessel differential equation of order $\frac{n-2}{2}$ whose solution is

$$v(\theta) = AJ_{\frac{n-2}{2}}(\theta) + BY_{\frac{n-2}{2}}(\theta) \tag{2.9}$$

where A and B are constants and $J_{\frac{n-2}{2}}(\theta)$ and $Y_{\frac{n-2}{2}}(\theta)$ are Bessel functions of order $\frac{n-2}{2}$ of first and second kind respectively. $Y_{\frac{n-2}{2}}(\theta)$ is singular at the origin, ie. $Y_{\frac{n-2}{2}}(\theta) \to \infty$ as $\theta \to 0$

Hence we choose $B = 0$ so that

$$v(\theta) = AJ_{\frac{n-2}{2}}(\theta)$$

or

$$v(|\varsigma|t) = AJ_{\frac{n-2}{2}}(|\varsigma|t) \qquad (2.10)$$

Now

$$J_n(|\varsigma|t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(n+r+1)} \left(\frac{|\varsigma|t}{2}\right)^{2r+n}$$

$$J_{\frac{n-2}{2}}(|\varsigma|t) = \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{|\varsigma|t}{2}\right)^{\frac{n-2}{2}} - \frac{1}{\Gamma(\frac{n-2}{2})}\left(\frac{|\varsigma|t}{2}\right)^{\frac{n-2}{2}}$$

Whose series expansion is

$$J_n(|\varsigma|t) = \frac{1}{\Gamma(n+1)}\left(\frac{|\varsigma|t}{2}\right)^{n} - \frac{1}{\Gamma(n+2)}\left(\frac{|\varsigma|t}{2}\right)^{n+2}$$

$$+\frac{1}{2!\Gamma(n+3)}\left(\frac{|\varsigma|t}{2}\right)^{n+4} + \ldots\ldots$$

$$+\frac{1}{2!\Gamma(\frac{n+2}{2})}\left(\frac{|\varsigma|t}{2}\right)^{\frac{n+6}{2}} + \ldots\ldots..$$

Thus (2.9) can be written as

$$v(|\varsigma|t) = AJ_{\frac{n-2}{2}}(|\varsigma|t) = A\left[\frac{1}{\Gamma(\frac{n}{2})}\left(\frac{|\varsigma|t}{2}\right)^{\frac{n-2}{2}} - \frac{1}{\Gamma(\frac{n-2}{2})}\left(\frac{|\varsigma|t}{2}\right)^{\frac{n-2}{2}} + \frac{1}{2!\Gamma(\frac{n+2}{2})}\left(\frac{|\varsigma|t}{2}\right)^{\frac{n+6}{2}} + \ldots\ldots..\right] \qquad (2.11)$$

and so,

$$\bar{u}(\varsigma,t) = At^{-(\frac{n-2}{2})}\left[\frac{1}{\Gamma(\frac{n}{2})}\left(\frac{|\varsigma|t}{2}\right)^{\frac{n-2}{2}} - \frac{1}{\Gamma(\frac{n-2}{2})}\left(\frac{|\varsigma|t}{2}\right)^{\frac{n-2}{2}} + \frac{1}{2!\Gamma(\frac{n+2}{2})}\left(\frac{|\varsigma|t}{2}\right)^{\frac{n+6}{2}} + \ldots\ldots..\right] \qquad (2.12)$$

as $t \to 0$, $\bar{u}(\varsigma,0) = A\left[\frac{1}{\Gamma(\frac{n}{2})}\left(\frac{|\varsigma|}{2}\right)^{\frac{n-2}{2}}\right] = F(\varsigma) = \bar{f}(\varsigma)$

thefore

$$A = \frac{\bar{f}(\varsigma)\Gamma(\frac{n}{2})}{(\frac{|\varsigma|}{2})^{(\frac{n-2}{2})}}$$

or

$$\bar{u}(\varsigma,t) = \frac{t^{-(\frac{n-2}{2})}\bar{f}(\varsigma)\Gamma(\frac{n}{2})}{(\frac{|\varsigma|}{2})^{(\frac{n-2}{2})}} J_{\frac{n-2}{2}}(|\varsigma|t)$$

$$\bar{u}(\varsigma,t) = 2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})\bar{f}(|\varsigma|)(|\varsigma|t)^{-(\frac{n-2}{2})} J_{\frac{n-2}{2}}(|\varsigma|t) \qquad (2.13)$$

$$\text{Now } \bar{u} = \int_{-\infty}^{\infty}\ldots\int \Gamma(\frac{n}{2})\int_{\mathbb{R}^n} \bar{f}(\varsigma)t^{-(\frac{n-2}{2})} J_{\frac{n-2}{2}}(\varsigma t)d\varsigma \qquad (2.13\text{ a})$$

We now write (2.13) in terms of the hypergeometric function.

Recalling the definition of the Bessel function

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(\nu+1; -\frac{z^2}{4}\right), \qquad (2.14)$$

we then let $\nu = \frac{n-2}{2}$, $n = 2(\nu+1)$ and $z = |\varsigma|t$ thus

(2.13) becomes

$$\bar{u}(z) = \left(\frac{2}{z}\right)^\nu \Gamma(\nu+1)\bar{f}(|\varsigma|)J_\nu(z). \qquad (2.15)$$

Now the hypergeometric part appearing in (2.14) has the following integral representation

$${}_0F_1\left(;\nu+1;-\frac{z^2}{4}\right) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\cosh(izt)dt;$$

$$\operatorname{Re}(\nu) > -\frac{1}{2} \qquad (2.16)$$

 ISSN: 2454-4248
13 – 19

We now test the convergence of ${}_0F_1\left(\nu+1;-\frac{z^2}{4}\right)$.

We know that

$$ {}_0F_1\left(\nu+1;-\frac{z^2}{4}\right)=\sum_{n=0}^{\infty}\frac{\left(-\frac{z^2}{4}\right)^n}{n!(\nu+1)_n}. $$

Hence by ratio test we have

$$ \lim_{n\to\infty}\left|\frac{\left(-\frac{z^2}{4}\right)^{n+1}}{(\nu+1)_{n+1}}\frac{(\nu+1)_n}{\left(-\frac{z^2}{4}\right)^n}\frac{n!}{(n+1)!}\right| $$

$$ =\lim_{n\to\infty}\left|\frac{\left(-\frac{z^2}{4}\right)}{(\nu+1)_{n+1}}\frac{(\nu+1)_n}{(n+1)}\right| $$

$$ =\lim_{n\to\infty}\left|\frac{\left(-\frac{z^2}{4}\right)}{(\nu+1)_n}\frac{(\nu+1)_n}{(\nu+1+n)(n+1)}\right| $$

$$ =\lim_{n\to\infty}\left|\frac{\left(-\frac{z^2}{4}\right)}{(n+1)}\frac{1}{(\nu+1+n)}\right|\to 0 $$

(2.17)

Now, $\frac{d}{dz}F\left(\nu+1;-\frac{z^2}{4}\right)=\frac{d}{dz}\sum_{n=0}^{\infty}\frac{\left(-\frac{z^2}{4}\right)^n}{n!(\nu+1)_n}$

$$ =\sum_{n=1}^{\infty}\frac{n\left(-\frac{z^2}{4}\right)^{n-1}}{n!(\nu+1)_n}=\sum_{m=0}^{\infty}\frac{\left(-\frac{z^2}{4}\right)^m}{m!(\nu+1)_{m+1}} $$

$$ =\frac{1}{(\nu+1)}\sum_{m=0}^{\infty}\frac{\left(-\frac{z^2}{4}\right)^m}{m!\{(\nu+1)+1\}_m} $$

$$ =\frac{1}{(\nu+1)_m}F\left(\{(\nu+1)+1\};-\frac{z^2}{4}\right) $$

(2.18)

In general,

$$ \frac{d^k}{dz^k}F\left(\nu+1;-\frac{z^2}{4}\right)=\frac{1}{(\nu)_k}\sum_{m=0}^{\infty}\frac{\left(-\frac{z^2}{4}\right)^m}{m!\{(\nu+1)+k\}}=\frac{1}{(\nu+1)_k}F\left(\{(\nu+1)+k\};-\frac{z^2}{4}\right),\qquad k=1,2,3... $$

(2.19)

(2.19) is thus infinitely differentiable hence it is a smooth function.

3. Inverse Fourier transform of $\bar{u}(\varsigma,t)$.

To complete the solution of (1.1) we have to determine the inverse Fourier transform of $\bar{u}(\varsigma,t)$.

$\bar{u}(\varsigma,t)$ is not an L^1 and hence we attempt to determine the inverse by use of the function

$$ g(x)=\left(t^2-x^2\right)^{\nu-\frac{1}{2}}H(t-x);\ \nu>\frac{1}{2} $$

(3.1)

where H is the Heaviside unit function defined by,

$$ H(t-x)=\begin{cases}1 & : x>t\\ 0 & : x\le t\end{cases}. $$

The multiplication of the even function $\left(t^2-x^2\right)^{\nu-\frac{1}{2}}$ by the Heaviside unit function $H(t-x)$ allows us to take the Fourier cosine transform of $g(x)$ as;

$$ \hat{g}_c(\varsigma)=\hat{f}_c(g(x))\ x\to\varsigma $$

$$ =\sqrt{\frac{2}{\pi}}\int_0^t\left(t^2-x^2\right)^{\nu-\frac{1}{2}}\cos\varsigma x\,dx $$

$$ =\sqrt{\frac{2}{\pi}}\int_0^t\left(t^2-x^2\right)^{\nu-\frac{1}{2}}\sum_{r=0}^{\infty}\frac{(-1)^r(\varsigma x)^{2r}}{\Gamma(2r+1)}dx $$

$$ =\sqrt{\frac{2}{\pi}}\sum_{r=0}^{\infty}\frac{(-1)^r(\varsigma)^{2r}}{\Gamma(2r+1)}\int_0^t x^{2r}\left(t^2-x^2\right)^{\nu-\frac{1}{2}}dx \quad (3.2) $$

where we may take t large enough.

Now let

$$ x^2=t^2u;\quad 2xdx=t^2du $$

Hence,

ISSN: 2454-4248
13 – 19

$$\hat{g}_c(\varsigma)=\sqrt{\frac{2}{\pi}}\sum_{r=0}^{\infty}\frac{(-1)^r(\varsigma)^{2r}}{\Gamma(2r+1)}\int_0^t t^{2r}u^r t^{2\nu-1}(1-u)^{\nu-\frac{1}{2}}\frac{t^2}{2tu^{\frac{1}{2}}}du$$

$$=\frac{t^{2\nu}}{\sqrt{2\pi}}\sum_{r=0}^{\infty}\frac{(-1)^r(\varsigma t)^{2r}}{\Gamma(2r+1)}\int_0^t u^{r-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}}$$

$$=\frac{t^{2\nu}}{\sqrt{2\pi}}\sum_{r=0}^{\infty}\frac{(-1)^r(\varsigma t)^{2r}}{\Gamma(2r+1)}B\left(\nu+\frac{1}{2},r+\frac{1}{2}\right) \quad (3.3)$$

But by duplication formula,

$$\Gamma\left(r+\frac{1}{2}\right)=\frac{\Gamma(2r+1)\Gamma\left(\frac{1}{2}\right)}{2^{2r}\Gamma(r+1)} \quad (3.4)$$

Hence,

$$\hat{g}_c(\varsigma)=\frac{t^{2\nu}}{\sqrt{2\pi}}\sum_{r=0}^{\infty}\frac{(-1)^r(\varsigma t)^{2r}}{\Gamma(2r+1)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{2^{2r}\Gamma(r+1)}$$

$$=\frac{1}{\sqrt{2}}t^{2\nu}\Gamma\left(\nu+\frac{1}{2}\right)\sum_{r=0}^{\infty}\frac{(-1)^r(\varsigma t)^{2r}}{\Gamma r!(\nu+r+1)} \quad (3.5)$$

The Bessel function $J_\varsigma(x)$ is defined as

$$J_\varsigma(x)=\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{\varsigma t}{2}\right)^{2r+\nu}}{\Gamma r!(\nu+r+1)} \quad (3.6)$$

Thus

$$\hat{g}_c(\varsigma)=\frac{t^{2\nu}}{\sqrt{2}}\Gamma\left(\nu+\frac{1}{2}\right)\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{\varsigma t}{2}\right)^{2r+\nu}}{r!\Gamma(\nu+r+1)}\cdot\left(\frac{2}{\varsigma t}\right)^{\nu}$$

$$=2^{\nu-\frac{1}{2}}\frac{t^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\varsigma^\nu}J_\nu(\varsigma t)$$

Hence,

(3.7

$$\sqrt{\frac{2}{\pi}}\int_0^t (t^2-x^2)^{\nu-\frac{1}{2}}\mathrm{H}(t-x)\cos(\varsigma x)dx$$

$$=2^{\nu-\frac{1}{2}}t^\nu\Gamma\left(\nu+\frac{1}{2}\right)\cdot\frac{J_\nu(t\varsigma)}{\varsigma^\nu} \quad (3.8)$$

i.e

$$\int_0^t (t^2-x^2)^{\nu-\frac{1}{2}}\cos(\varsigma x)dx$$

$$=\sqrt{\frac{\pi}{2}}2^{\nu-\frac{1}{2}}t^\nu\Gamma\left(\nu+\frac{1}{2}\right)\cdot\frac{J_\nu(t\varsigma)}{\varsigma^\nu} \quad (3.9)$$

The inversion formula gives,

$$2^{\nu-\frac{1}{2}}t^\nu\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{J_\nu(\varsigma t)}{\varsigma^\nu}\cos(\varsigma t)dx$$

$$=(t^2-x^2)^{\nu-\frac{1}{2}}H(t-x) \quad (3.10)$$

Therefore,

$$\int_0^t\frac{J_\nu(t\varsigma)}{\varsigma^\nu}\cos(\varsigma x)dx=\frac{\sqrt{\frac{\pi}{2}}(t^2-x^2)^{\nu-\frac{1}{2}}H(t-x)}{\Gamma\left(\nu+\frac{1}{2}\right)2^{\nu-\frac{1}{2}}t^\nu};$$

$$\nu>-\frac{1}{2} \quad (3.11)$$

Now from the above theorem, and taking

$f(x)=\delta,$

$\bar{f}(\varsigma)=\bar{\delta}=1$,

and

$n-1=2m+1$

(2.13a) can be written as

$$\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{2}t^{2\nu}}\left\{2^{\nu-\frac{1}{2}}t^\nu\Gamma\left(\nu+\frac{1}{2}\right)\cdot\frac{J_\nu(t\varsigma)}{\varsigma^\nu}\right\} \quad (3.12)$$

Hence, on comparing (3.8) and (3.9) we claim that the inverse Fourier Transform of the equation
(2.13a)

subject to the initial conditions $u(x.0) = f(x) = \delta(x)$ and $u_t(x,0) = 0$ on the range $[0,\infty)$ is given by the function

$$g(x) = (t^2 - x^2)^{\nu-\frac{1}{2}} H(t-x) \text{ i.e}$$

$$= \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{2}t^{2\nu}} \square 2^{\nu-\frac{1}{2}} t^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)\cdot\frac{J_\nu(t\varsigma)}{\varsigma^\nu}$$

i.e

$$\int_0^t (t^2 - x^2)^{\nu-\frac{1}{2}} \cos(\varsigma x)\,dx$$

$$= \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{2}t^{2\nu}} \sqrt{\frac{\pi}{2}}\, 2^{\nu-\frac{1}{2}} t^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)\frac{J_\nu(t\varsigma)}{\varsigma^\nu}$$

$$= \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{t^{2\nu}} \sqrt{\pi}\, 2^{\nu-\frac{3}{2}} t^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)\frac{J_\nu(t\varsigma)}{\varsigma^\nu}$$

$$\sqrt{\frac{2}{\pi}}\int_0^t (t^2 - x^2)^{\nu-\frac{1}{2}} \mathrm{H}(t-x)\cos(\varsigma x)\,dx$$

Hence $u(x,t) = (t^2 - x^2)^{\nu-\frac{1}{2}} H(t-x)$ solves the EPD

(3.13)

CONCLUSION

Equation (3.13) represents the solution of (1.1) subject to the conditions (1.2).

REFFERENCES

[1] Weinstein A (1952).*On the Cauchy problem for the Poisson equation and the wave equation.* C. R. Acad. Sci., 234: 2584-2585.

[2] Wanjala Iyaya CC, Manyonge WA, Esekon J (2012*). On singular Cauchy problem of Euler-oisson Darboux equation*. Pioneer J. Math. Math. Sci., 5(1): 121-136.

[3] Hadamard, J. (1923), *Lectures on the Cauchy's Problem in Linear Partial Differential Equations,*Yale University Press, New Haven, recent edition: Nabu Press, 2010,ISBN 9781177646918

[4] E. K. Blum (1954), The Euler-Poisson-Darboux equation in the exceptional cases . American Mathematical Society ; DOI: 10.2307/2032026

[5] IyayaWanjala and Isaac Chepkwony (2015.) *Solutions of Generalized Euler-Poisson-Darboux-Equation and their Properties*. Journal of Physical Science and Environmental

[6] Seilkhanova RB, Hasanov AH (2015). Particular solutions of generalized Euler-Poisson-Darboux equation. Elect. J.Diff. Equat., 09: 1-10.

[7] Copson,E.T (1958).*On the Riemann-Green function.* Acive for Rational Mechanics and Analysis.

[8] Darboux, G.(1915).*lecons sur la the orie ge nerale des surfaces*. Second edition. Gautheravillas. Paris.

[9] Poisson, S.(1823).*memoirs suri'inte'gration des e'quate ons line'aires aux derive'es partielles,* jour. Ecole Polytech, vol 12.

[10] Iyaya,W. and Manyonge,M. (2010) *Singular Cauchy Problem of Euler-Poisson-Darboux Equation* , International Journal of Mathematics and Mathematical Sciences.

[11] A. Manyonge, D. Kweyu, J. Bitok, H. Nyambane and J. Maremwa(2013) ,*Weak Solution of the Singular Cauchy Problem of Euler-Poisson-Darboux Equation*. Applied journal Mathematical Sciences, Vol. 7, no. 7, 315 – 325.

[12] Euler, L: *Foundations of integral calculus* vol 3, Academia Scientiarum Imperialis Petropolitanae, 1770

[13] Iyaya et al (2018), *Applying Transmutation method to solve Euler-Poisson –Darboux Equation,* Pioneer Journal of Mathematics and Mathematical sciences. ISSN 2230-9829.