Vectors Space and Module

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ABSTRACT: A vector space is defined as a set that is near under limited vector expansion and scalar augmentation. It has scalars with increase by rational numbers, complex numbers or any other track. In abstract algebra, the fundamental algebraic structure is called as module. In this vector space and module to be same in terms of definition. Module has coefficients in much more general algebraic objects. But when we analysed deeply, they are quite different as discussed in this paper.

***Key words:** Vector space, linear space, scalars, module, field, abstract algebra.*

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I. INTRODUCTION

Vector spaces owe their significance to the way that such huge numbers of models emerging in the arrangements of explicit issues end up being vector space.

In a vector space, the arrangement of scalars is a field and follows up on the vectors by scalar augmentation, subject to specific aphorisms, for example, the distributive law.

Infinite dimensional vector spaces emerge normally in numerical investigation, as function spaces, whose vector are functions. These vector spaces are commonly blessed with extra structure, which might be a topology, permitting the thought of issues of vicinity and progression.

This is especially the situation of Banach spaces and Hilbert spaces, which are major in scientific examination.

The coefficients used for linear combinations in a vector space are in a field, but there are many places where we naturally meet linear combinations with coefficients in a ring.

A module in linear algebra, the ultimate imperative architecture is that of a vector space over a area. For commutative algebra it is in this manner helpful to acknowledge the speculation of this idea to the situation where the basic space of scalars is a commutative ring R rather than a field. The subsequent structure is known as a module.

Actually, there is another progressively unobtrusive motivation behind why modules are exceptionally amazing. They bind together numerous different structures that you definitely know.

Therefore, general outcomes on modules will have various results in a wide range of setups. So given us now a chance to begin with the meaning of modules. On a fundamental level, their hypothesis that we will at that point rapidly talk about in this part is totally similar to that of vector spaces.

These comments will turn out to be clear when we make the meaning of a vector space. Vector spaces owe their significance to the way that such a large number of models emerging in the arrangements of explicit issues end up being vector spaces.

Thus the fundamental ideas presented in them have a specific comprehensiveness and are ones we experience, and continue experiencing, in such huge numbers of assorted settings.

Among these principal ideas are those of linear dependence, premise and measurement which will be created in this part.

In the last piece of present part we sum up from vector spaces to modules; generally, a module is a vector space over a ring rather than over a field. For limitedly produced modules over Euclidean rings we will demonstrate the central premise theorem.

For limitedly produced modules over Euclidean rings we will demonstrate the central premise theorem. This outcome enables us to give a total depiction and development of all abelian bunches which are produced by a limited number of components.

A module is a vector space over a ring instead of over a field.

However, although many properties just carry over without change, others will turn out to be vastly different.

Of course, proofs that are literally the same as for vector spaces will not be repeated here; instead we will just give references to the interrelated popular linear algebra allegation in these fields.

The objective of this dissertation is to study an application of Vector space and module.

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This dissertation presents the salient aspects of the subject matter in five chapters. Definitions, Lemmas, Theorems and Corollaries are numbered serially in this dissertation.

BASIC DEFINITION AND EXAMPLES

Definition 1.1

A non-empty set V is called a **vector space** over F , if

(i) V is an abelian group, for an operation denoted by '+'

(ii) (ii) For each $a \in F$ and $v \in V$, an element $av \in V$ is defined, satisfying the following conditions

$$a(v+v') = av + av', a \in F, v, v' \in V$$

$$(a+b)v = av + bv, a, b \in F, v \in V$$

$$a(bv) = (ab)v, a, b \in F, v \in V$$

$$1v = v, v \in V$$

Definition 1.2

It is consider that V defined be a vector space over F and $W \subset V$ .Then W is known a **subspace** of V if

(i) W Is defined as a subspace of the abelian group V .

(ii) each $a \in F$ and $w \in W$, $aw \in W$

It is obvious from the definition that W is shut for expansion and scalar duplication and that the maxims for a vector space are fulfilled .Thus W is additionally a vector space over F .

Definition 1.3

For any subset S of V, the convergence of all subspaces of V consists S is known as the subspace created by S. It is typically indicated by the image $L(S)$.

Definition 1.4

It is consider that V work as a vector space over F and let V_1, V_2 be known as subspaces of V .Then V is known as a **direct sum** of V_1 and V_2 if

$V = V_1 + V_2$ and

$V_1 \cap V_2 = \{0\}$.

Definition 1.5

It is consider that V be a work as vector space and $V_1, V_2 \ldots V_n$ be known as subspaces of V .Then V is a direct sum of $V_1, V_2 \ldots V_n$ if every v € V can be particularly noticed as $V = V_1 + V_2 + \ldots \ldots + V_n$ .Where v_i € V_i

We use the notation V= $V_1 \oplus V_2 \oplus \ldots \oplus V_n$ to indicate that V is considered as a direct sum of the subspaces $V_1, V_2 \ldots V_n$.

SOME PROPOSITIONS ON VECTOR SPACES AND MODULES

PROPOSITION 2.1

It is consider that V is used as a vector space over F and $\{W_\alpha\}$ $\alpha \in I$ a collection of subspaces of V .Then w = $\bigcap_{\alpha \in I} W_\alpha$ is also a subspace of V .

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Proof:

Let $v \in W$ and $v^1 \in$ W.

Then v and $v^1 \in W_\alpha$ for each $\alpha \in I$.

Since W_α is a subspace

We have,

$v\text{-}v^1 \in W\alpha$ and $av \in W_\alpha$ for each a € I.

As this is correct for each α ,

We have,

$v\text{-}v^1 \in W$ and $av \in W$.

Hence W consider as a subspace of V .

PROPOSITION 2.2

It is consider that V will be a vector space over F and S a sub set of V . Then L(s) = $\{v \mid v = \sum_{i=1}^{n} a_i v_i$ $a_i \in F, v_i \in S, n \geq 1\}$

Proof:

Let $W = \{v \mid v = \sum_{i=1}^{n} a_i v_i \;\; a_i \in F, v_i \in S, n \geq 1\}$

We shall show first that,

W is recognized as subspace of V

Let v, v' € W

So that $v = \sum_{1}^{n} a_i v_i \;\; a_i \in F, v_i \in S$

And $v' = \sum_{1}^{m} b_j v_j' \;\; b_j \in F, v_j' \in S$

Then,

$$v - v' = \sum_{1}^{n} a_i v_i - \sum_{1}^{m} b_j v_j' = \sum_{1}^{n} a_i v_i + \sum_{1}^{m} (-b_j) v_j' \in W .$$

Also $av = \sum_{1}^{n} (a \; a_i) \; v_\text{i} \in W$ for any $a \in F$.

Thus W is recognized as a subspace of V.

Since for any *v € S, v=1v,*

We have,

v € W.

Hence,

S $\subset$ W and so L(S) $\subset$ W.

Since L(S) is the minimum subspace consist of *S*.

Consider that w_1 be each subspace of V consists of S.

Then for *v € W, v=* $\sum_{1}^{n} a_i v_i$, v_i *€ S.*

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Hence $v \in W^1$ as each $v_i \in S \subset W^1$ and W^1 is a subspace.

Thus,

$W \subset W^1$ for any subspace W^1 containing S.

This implies that $W \subset L(S)$ and

Hence $W = L(S)$

Hence the proof.

SOME THEOREMS ON VECTOR SPACES AND MODULES

Theorem 3.1

It is consider that V is the vector spaces of dimensions *m* and V' it the vector spaces of dimensions *n* over *F*. Later dim L (V, V′) =*mn*

Proof*:*

Let $\{v_1, v_2 \ldots, v_m\}$ be any basis of V and

Let $\{v'_1, v'_{2, \ldots\ldots} v'_n\}$ any basis of V'.

We define *mn* linear transformation,

$T_{ij}: V \to V'$, $1 \le i \le m, 1 \le j \le n,$

By the conditions,

$T_{ij}(v_i) = vj$, and $T_{ij}(v_k) = 0, i \ne k$.

Each T_{ij} is well defined for each *i,j*

Because it is decided by its values on the base elements $\{v_1, v_2 \ldots, v_m\}$

These values can be prescribed arbitrarily.

We shall show that,

The *mn* elements $\{T_{ij}\}$ form a basis of *L (V, V′)*.

They form a generating set,

Because for any $T \in L(V, V')$

Let $T(v_i) = \sum_{j=1}^{n} a_{ij} v'_j, a_{ij} \in F$

Consider the linear transformation

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} T_{ij}$$

Now $(\sum_i \sum_j a_{ij} T_{ij})(v_i) = \sum_{j=1}^{n} a_{ij} v'_j = T(v_i)$ for all *i*.

Thus the linear mappings T and $\sum_{ij}^{n} a_{ij} T_{ij}$ have identical values on the basis elements $\{v_1, v_{2, \ldots\ldots} v_m\}$.

Hence $T = \sum_{ij} T_{ij}$.

To show that,

$\{T_{ij}\}$ are linearly independent,

Consider a linear relation of the type

$$\sum_{ij}^{n} b_{ij}T_{ij} = 0, b_{ij} \in F .$$

Taking the values on V_l

We have,

$$\sum_{j=1}^{n} b_{ij} v'_j = 0, \text{ for all } l, 1 \le l \le m$$

Since $\{v'_1, v'_{2\ldots\ldots} v'_n\}$are linearly independent,

We have,

$$b_{ij} = 0 \text{ for all } j.\ 1 \le j \le n, \text{ and all } l.\ 1 \le l \le m .$$

Thus,

$\{T_{ij}\}$ form a basis of $L(V, V^1)$ showing that

dim L (V, V^1) $= mn$

Hence the proof.

Theorem 3.2

Let $f : M \to N$ be a homomorphism of M onto N where M and N are R-modules.

Then the kernel of $f = \{x \in M \mid f(x) = 0\}$ is a sub module K of M and the quotient module $M \mid K$ is isomorphic to N.

Proof:

It is easy to verity that,

K satisfies the conditions for a sub module.

The map $f : M \mid K \to N$ given by

$$f(x + K) = f(x), x \in M$$

It is well defined and

It is an isomorphism of R-modules.

Hence the theorem.

II. CONCLUSION

In this course we have discovered that advanced algebra is an investigation of sets with tasks characterized on them. As the fundamental model we have begun a precise investigation of groups. Groups theory is a standout amongst the most critical territories of contemporary science, with applications running from chemistry and physics to cryptography and coding. However from the definition point of view, the concept of vector space and Modules looks like similar. Lots of properties as discussed in this paper differ due to this reason. Results of vector spaces are very much different from the results of modules as discussed in paper. So, the roles of scalar in vector space and modules play a very important role which creates drastic changes in the results.

REFERENCES

[1]. Brown, William A.Matrices and vector spaces, New york M, 1991. Dekker, ISBN 978-8247-8419-5
[2]. Lang, Serge. Linear algebra, Berlin, New York: springer – verlag, 1987. ISBN 978-0-387-96412-6
[3]. Roman, striven. Advanced Linear Algebra, Graduate Texts in mathematics, 135 (2nd ed.), Berlin, New York: Springer – verlag, 2005. ISBN 978-0-387-24766-3.
[4]. Nathan Jacobson. Structure of rings. Colloquium publications, 2nd Ed., AMS Bookstore, 1964, 37. ISBN 978-0-8218-1037-8.
[5]. Hazewinkel, Michiel, ed. Module encyclopedia of Mathematics, Springer, 2001. ISBN 978-1-55608-010-4.
[6]. Meyer, Carl D. Matrix Analysis and Applied Linear Algebra, Society for industrial and Applied Mathemathics (SIAM), 2001. ISBN 978-0-8987-454-8[dead link]
[7]. Leon, Steven J.Linear Algebra with Applications (7th ed.), Pearson Prentice Hall, 2006.
[8]. Anderson FW, Fuller KR.Rings and categories of modules, graduate texts in mathematics, 2nd Ed., Springer-Verlag, New York, 1992.