The Fundamental Group on Algebraic Topology

B. Senthilkumar
M.Sc., M.Phil., B.Ed.,
Assistant Professor, Department of Mathematics,
PRIST University, Vallam, Thanjavur

S. Govindasamy
M.Phil - Research Scholar, Department of Mathematics,
PRIST University, Vallam, Thanjavur

ABSTRACT: Algebraic topology is mostly about finding invariants for topological spaces. The fundamental group is the simplest, in some ways, and the most difficult in others. This project studies the fundamental group, its basic properties, some elementary computations, and a resulting theorem.

In this paper, we will examine the construction and nature of the first homotopy group, which is more commonly known as the fundamental group of a topological space. We will first briefly cover the basics of point-set topology, then use these concepts to facilitate a rigorous study of the construction of the fundamental group.

Keywords: *Fundamental group, algebraic topology, homotopy, topological space*

*****

I. INTRODUCTION

The subject of topology is of interest in its own right, and it also serves to lay foundations for future study in Analysis, in Geometry and in Algebraic Topology.

There is an idea more general than the idea of simple connectedness, an idea that includes the fundamental group of the space. Two space that homeomorphic have fundamental groups that are isomorphic. And the condition of simple connectedness is just the condition that the fundamental group of X is trivial (one element) group.

We define the fundamental group and study its properties. Then we apply it to a number of problems, including the problem of showing that various spaces, such as those already mentioned, are not homeomorphic.

Several problem of Topology are converted to algebraic problems though Algebraic Topology. In some cases the solutions may be easier to find than that of the corresponding problems in Topology.

The present project being an introductory and elementary in nature, deals with several basic ideas of Algebraic Topology.

PRELIMINARIES

Definition: 1.1

A topology on the set X is a collection τ of subsets of X having the following properties.

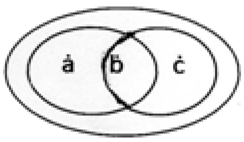
1. X and ϕ are in τ
2. The union of the element of any sub-collection of τ is in τ.
3. The intersection of elements of any finite sub collection of τ is in τ.

The set X for which a Topology τ has been specified is called a **Topological space**.

Example: 1.1.1

Let X be a three-element set, $X = \{a, b, c\}$

$\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$



(X, τ) is a Topological space.

Definition: 1.2

If X is a topological space with topology τ, we say that subset U of X is an **open set** of X if U belongs to the collection τ.

Example: 1.2.1

Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$

ISSN: 2454-4248
22 – 26

The sets $\{a\}, \{b\}, \{a, b\}$ are open sets.

Definition: 1.3

A subset "A" of a topological space X is said to be **closed** if a set $X - A$ is open.

Example: 1.3.1

Let $X = \{a, b, c, d\},\ \tau = \{X, \phi, \{a\}, \{a, c\}, \{a, b, d\}\}$

The set $\{b, d\}, \{c\}, \{b, c, d\}$ are closed sets.

Definition: 1.4

Asubset "A" of the topological space X, the **closure of A** is defined as the intersection of all closed sets containing A. The closure of A is denoted by $\bar{A}$.

Obviously $\bar{A}$ is a closed set $A \subset \bar{A}$.

Example: 1.4.1

Let $X = \{a, b, c, d\}$

$\tau = \{X, \phi, \{a\}, \{b, c\}, \{a, d\}\{a, b, c\}\}$

If $A = \{b\}, B = \{a, b\}, C = \{b, c, d\}$ then the closed subsets are

$X, \phi, \{b, c, d\}\{b, c\}, \{a, d\}, \{d\}$.

$$cl(A) = cl(\{b\}) = \{b, c\}$$

$$cl(B) = cl(\{a, b\}) = X$$

$$cl(c) = cl(\{b, c, d\}) = \{b, c, d\}$$

It is noted that,

If A is Open then A = Int A and

If A is Closed $A = \bar{A}$

Definition: 1.5

Let A and B be two non empty sets and if some way there corresponds to each $X \in A$, an unique element $Y \in B$ then the correspondence is called a mapping of A to B and is denoted by

$$f: A \to B$$

Where A is called **domain** and B is called **co-domain**.

HOMOTOPY OF PATHS

We now consider paths on X and an equivalence relation called **path homotopy** between them. And we shall define a certion operation on the collection of the equivalence classes that makes it into what is called in algebra a groupoid.

Definition: 2.1

If f and f' are continuous maps of the space X into the space Y, then f is **homotopic** to f' if there is a continuous map $F: X \times I \to Y$ such that

$F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for each x.

(Here $I = [0,1]$)

The map F is called **homotopy** between f and f'.

We also say that x_o is the **initial point** and x_1 is the **final point.**

Definition: 2.2

An equivalence relation on the set A is a relation C on A having the following three properties.

i) Reflexivity

$x\ C\ x$ for every x in A

ii) Symmetry

If $x\ C\ y$, then $y\ C\ x$

iii) Transitivity

If $x\ C\ y$ and $y\ C\ z$ then $x\ C\ z$.

Theorem: 2.3

The relation $\simeq$ and $\simeq$ p are equivalence relations.

Proof:

Let us verify the properties of an equivalence relation.

i) $\boldsymbol{x\ R\ x}$ implies $\boldsymbol{f\ R\ f}$

Given f is continuous, hence it is trivial that $f\ R\ f$.

The map $F(x, 0) = f(x)$ and $F(x, 1) = f(x)$

ISSN: 2454-4248
22 – 26

(i.e) $F(x,t) = f(x), t \in [0,1]$

F is a path having same end points.

If f is a path, F is a path homotopy.

ii) $f \simeq f'$ implies $f' \simeq f$

Given $f \simeq f'$

$F(x,0) = f(x)$

$F(x,1) = f'(x)$

To Prove $f' = f$

$G(x,0) = f'(x)$

$G(x,1) = f(x)$

(i.e) $G(x,t) = F(x,1-t)$ is a homotopy between f' and f.

F is a path homotopy, so is G.

iii) $f \simeq f'$ and $f \simeq f''$ implies $f \simeq f''$

Let F be a homotopy from f to f'

Let F' be a homotopy from f to f''

(i.e) $F(x,0) = f(x)$ and $F(x,1) = f'(x)$

Also

$F'(x,0) = f'(x)$ and $F'(x,1) = f''(x)$

Now we define

$G: X \times I \to Y$ such that

$$G(x,t) = \begin{cases} F(x,2t) \; for \; t \in \left[0,\frac{1}{2}\right] \\ F'(x,2t-1) \; for \; t \in \left[\frac{1}{2},1\right] \end{cases}$$

A map G is well defined, since if $t = \frac{1}{2}$.

We have $F(x,2t) = f'(x) = F'(x,2t-1)$

Because G is a continuous on the two closed subsets

$X \times \left[0,\frac{1}{2}\right]$ and $X \times \left[\frac{1}{2},1\right]$ of $X \times I$,

It is continuous by pasting lemma.

Thus G is path homotopy.

Hence a path homotopy is equivalence relation.

THE FUNDAMENTAL GROUP

The set of path – homotopy classes of paths in a space X does not from a group under the operation $*$ because the product of two path – homotopy classes is not always defined.

But suppose we pick out a point x_o of X to serve as a "base point" and restrict ourselves to those paths that begin and end at x_o.

The set of these path-homotopy classes does from a group under $*$. It will be called **Fundamental Groupof X.**

In this chapter, we study the fundamental group and derive some of its properties.

In particular, we shall show that group is a topological invariant of the space X, the fact that is of crucial importance in using it to study homeomorphism problems.

Definition: 3.1

Let us consider G and G' be two groups, written multiplicatively. A **homomorphism** f: G→G' is a map such that $f(x,y) = f(x).f(y)$ for all x, y.

it automatically satisfies the equations $f(e) = e'$ and $f(x^{-1}) = f(x)^{-1}$.

Where e and e' are the identities of G and G' respectively and the exponent -1 denotes the inverse.

Definition: 3.2

The kernel of f is the set $f^{-1}(e')$.

It is a **subgroup** of G.

Definition: 3.3

Let X be a space; let x_o be a point of X. A path in X that begins and ends at x_o is called a **loop based** at x_o.

The set of path homotopy classes of loops based at x_o, with the operation $*$, is called the **fundamental group** of **X** relative to the base point x_o .

 ISSN: 2454-4248
22 – 26

It is denoted by $\Pi_1(X,x_o)$.

Example: 3.1

Let R^n denote the Euclidean n-space.

Then $\Pi_1(R^n, x_o)$ is a trivial group.

For if f is a loop in R^n based at x_o, the straight – line homotopy is a path homotopy between f and the constant path at x_o.

More generally, if X is any convex subset of R^n

Then $\Pi_1(X,x_o)$ is the trivial group.

In particular,

The unit ball B^n in R^n.

$B^n = \{X/x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2 \leq 1\}$,

has trivial fundamental group.

Definition: 3.4

Let $\propto$ be a path in X from x_o to x_1 we define a map,

$\hat{\propto}: \Pi_1: (X, x_o) \rightarrow \Pi^1(X, x_1)$

By the equation

$\hat{\propto}([f]) = [\propto] * [f] * [\propto]$

The map $\hat{\propto}$, which we call **"∝-hat"** is well-defined because the operation $*$ is well-defined.

If f is a loop based at x_o, then $\bar{\propto} * (f * \propto)$ is a loop based at x_1.

Hence $\hat{\propto}$ maps $\Pi_1(X, x_o)$ into $\Pi_1(X, x_1)$ as desired; note that it depends only on the path-homotopy class of $\propto$.

It is given in Figure 3.4

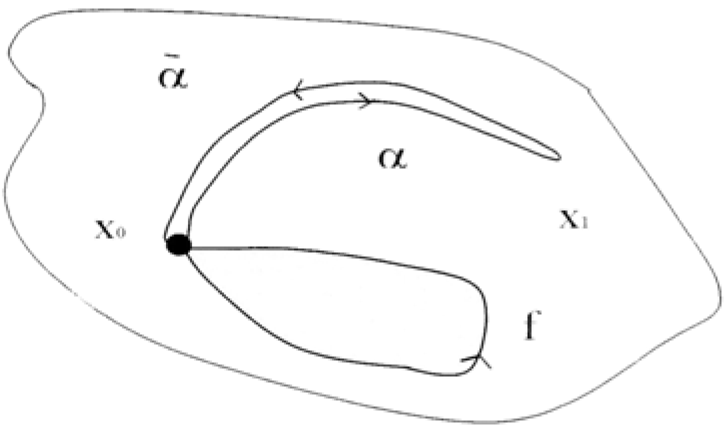


Figure 3.4

Theorem: 3.1

The map $\hat{\propto}$ is a group isomorphism.

Proof:

We consider,

$$\hat{\propto}([f]) * \hat{\propto}([g]) = ([\bar{\propto}] * [f] * [\propto]) * ([\bar{\propto}] * [g] * [\propto])$$
$$= [\bar{\propto}] * [f] * [g] * [\propto]$$
$$= \hat{\propto}([f] * [g]).$$

To show that,

$\hat{\propto}$ is an isomorphism.

We show that,

If β denotes the path $\bar{\propto}$,

Which is the reverse of $\propto$.

Then $\hat{\beta}$ is inverse of $\hat{\propto}$.

We compute for each element $[h]$ of $\Pi_1(X, x_1)$

$\hat{\beta}([h]) = [\bar{\beta}] * [h] * [\beta] = [\propto] * [h] * [\bar{\propto}]$

 ISSN: 2454-4248
22 – 26

$$\hat{\propto}\,(\hat{\beta}([h])) = [\overline{\propto}] * ([\propto] * [h] * [\overline{\propto}]) * [\propto] = [h]$$

a similar computation shows that

$$\hat{\beta}\big(\hat{\propto}\,([f])\big) = [\bar{\beta}] * \big([\beta] * [f] * [\bar{\beta}]\big) * [\beta] = [f] \text{ for each } [f] \in \Pi_1(X, x_o)$$

Hence the proof.

II. CONCLUSION

In this article, we introduced a fundamental group for a topological space. Though the idea of this group is not entirely complicated. In this paper, we define the singular homology and compute the homology groups of some certain spaces. We have seen that the homology is a functorial property.

So that it can be used to distinguish and classify the spaces. There are still more results to be found calculating the fundamental group of new transformation groups.

REFERENCES

[1]. **BALACHANDRAN V.K** Topological Algebras, Narosa Publishing House,
[2]. New Delhi (1999).
[3]. **DUGUNDJI.J** Topology, Allyn and Bacon, Boston (1966).
[4]. **GRAY.B Homotopy** theory, Academic Press (1975).
[5]. **KELLEY.J.L** General Topology, Springer- Verlag New York, (1991).
[6]. **LAHIRI.B.K**. A First Course In algebraic topology, Narosa Publishing
[7]. House (2000).
[8]. **MUNKRES JAMES.R** Topology, Pearson Education (2002).