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Non-Abelian Transformations in SU (2), SU (3) and Discrete Symmetries

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***Abstract* —** The pedagogical introduction about the non-abelian transformations and their application in SU(2), SU(3) symmetries is presented. Chiral Gauge groups and discrete symmetries like parity and charge congjugation are also discussed.

Keywords-*Non-Abelian transformations,Gauge field,QCD, discrete symmetries*

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I. INTRODUCTION

The abelian transformations as for electrodynamics depend on a single parameter ξ. Theyare invariant under local phase transformationsandare commuting. That means if the order of the transformation is changed, outcome is the same. Thenon-abelian transformations, on the other hand, depend on several parameters and do not necessarily commute. Therefore the outcome of twoconsecutive transformations depends on the orderin which the transformations are performed.The generators of these transformations, as we shall see below, also do not commute.

Further the number of parameters, on which anon-abelian gauge transformationdepends, indicates the number of independent gauge fields.Thereforethe gauge fields constitute the so-called representation group, whose member elementstransform among them. This is similar to the componentsof a three-dimensional vector which transform among themselves under thegroup of rotations. These gauge fields are self-interacting as the matricesrepresenting these fields do not commute.Let some fields are represented by matrices U which are elements of a group G.These matrices are expressible in terms of generators $t_a$which are said to obey the following equation, $[t_a, t_b] = f_{ab}^c t_c$ withnon-zero f_{ab}^c .These commutationrelations are known as the Lie algebra and the corresponding group G which follows this algebra is called a Lie group. The proportionality constants f_{ab}^c are called the structure constants,because they define the multiplication properties of the Lie group G. The expression $[t_a, t_b] \neq 0$ indicates the non abelian nature of the group elements(Please see Ref.[1-7] for detail).

II. NON-ABELIAN TRANSFORMATION

The group transformation$U$ transforms the fields $\psi(x)$as follows

$$\psi(x) \to \psi'(x) = U\psi(x) \tag{1}$$

Here $\psi(x)$ in general, denotes an array of different field components written as a column vector. In terms of components, we may write

$$\psi_i(x) \to \psi_i'(x) = U_{ij}\,\psi_j(x) \tag{2}$$

The matrices U can be written in exponentialform

$$U = \exp(\xi^a t_a) \tag{3}$$

where the matrices $t_a$are the generators of the representation of the group and the ξ^a constitute a set of linearly independentreal parameters in terms of which the group elements are described.

The dimension of the group is the number of generators,which is obviously equal to the number ofindependent parameters ξ^a.However, dimension of the group is unrelatedto the dimension of the matrices U and t_a. It is often straightforwardto determine the dimension or the number of generators of a group. For example, the generators of the SO(N) group areN × N real and anti-symmetricmatrices, in order that Eq.(1) defines an orthogonal matrix,$U^T = U^{-1}$. Therefore total number of elements possible is $N^2 - N$ as N is the diagonal elements. Due to condition of anti-symmetricity, $\frac{1}{2}N(N-1)$ independent matrices are possible. Therefore the dimensionof the SO(N) group is equal to $\frac{1}{2}N(N-1)$. The SU(N) group with the defining relation $U^{\dagger} = U^{-1}$ requires the generators $t^a$to be anti-hermiteanN×Nmatrices. Furthermore, member matrices U should haveunit determinant. As we know, anti-hermitean matrices t^a are traceless matrices. ThereforeN^2-1 independent anti-hermitean traceless matrices are possible and so the dimensionoftheSU(N) group is equal to $N^2 - 1$.

Photon or more precisely the photon field is electrically neutral and therefore does not couple to itself. However, these non-commuting gauge fieldsare not electrically neutral and interact with each other. Here we shall discuss the invariance of Lagrangian under non-abelian transformations.

Let us consider an infinitesimal transformations, where the parameters ξ^aaresmallso that from Eq.(3), we have

$$U = 1 + \xi^a t^a + O(\xi^2). \tag{4}$$

Because the matrices U defined in Eq.(1) constitute a representation of thegroup, these matrices must be closed under multiplication and so should be expressible into the same exponentialform. This leads to an important condition on the matrices t^a that their commutator is a linear combination of the same set of generatorsi.e.

$$[t_a, t_b] = f_{ab}^c t_c \tag{5}$$

Differentiating Eq.(1), we see

$$\partial_\mu \psi(x) \to \left(\partial_\mu \psi(x)\right)' = U(x)\partial_\mu \psi(x) + \left(\partial_\mu U(x)\right)\psi(x) \tag{6}$$

Therefore$\psi(x)$ transform according to Eq.(1) i.e. $\psi(x) \to \psi'(x) = U\psi(x)$ and such quantities are called as covariant quanties. However, due to the presence of the second term on the right-hand side of Eq.(6), derivative $\partial_\mu \psi(x)$ does not transform covariantly. Therefore ∂_μneeds to be replaced by a covariant quantity called covariant derivative D_μ. The transformation rule for D_μ should be

$$D_\mu \psi(x) \to \left(D_\mu \psi(x)\right)' = U(x) D_\mu \psi(x) \tag{7}$$

As we know for abelian transformations [see for example 1, 2].

$$\psi(x) \to \psi'(x) = e^{iq\xi}\psi(x) \tag{8}$$

Comparing with Eq.(1) we can say here the transformation $U = e^{iq\xi}$. Although U is abelian here and is called the phase transformation which generates a group of 1×1 unitary matrices called U(1). It has only one parameter q which measures the strength of the transformation. The covariant derivative for this abelian transformation is given by

$$D_\mu \equiv \partial_\mu - iqA_\mu \tag{9}$$

WhereA_μ follows a transformation rule of

$$A_\mu \to \acute{A}_\mu = A_\mu + \partial_\mu \xi \tag{10}$$

Where∂_μ is the ordinary derivative, is the called the coupling strength and A_μ is called the gauge field for the transformation. The infinitesimal displacement ξ is called a covariant translation. This formula for covariant derivative D_μ can be generalized for the covariant derivative for an arbitrary group. Namely, we takethelinear combination of an ordinary derivative and an infinitesimal gaugetransformation, where the parameters of the latter define the non-abelian gauge fields.Therefore, for a non-abelian gauge field W_μ , the covariant derivative can be written as

$$D_\mu \psi \equiv \partial_\mu \psi - W_\mu \psi \tag{11}$$

W_μ has the characteristic feature of a gauge field, as it carries the information regarding the group from one space-time point to another.

Here W_μ is represented by a matrix which is a member of a Lie Group Gand therefore, can be expressed as a linear combination of its generators t_a.

$$W_\mu = W_\mu^a t_a \tag{12}$$

Let us transform $W_\mu \psi$ under gauge transformation as defined by Eq.(6) as

$$\begin{aligned}(W_\mu \psi)' &\equiv \partial_\mu \psi' - (D_\mu \psi)' \\ &= U(\partial_\mu \psi) + (\partial_\mu U)\psi - U(D_\mu \psi) \\ &= \{U W_\mu U^{-1} + (\partial_\mu U) U^{-1}\}\psi'\end{aligned}$$

Therefore, thetransformation rule for W_μ is

$$W_\mu \to (W_\mu') \equiv U W_\mu U^{-1} + (\partial_\mu U) U^{-1} \tag{13}$$

Clearly the gauge field W does not transform covariantly but it defines a group. Successiveapplication of two transformations U_1 and U_2give $U_3 = U_1U_2$,whereU_3 is another member of the group. To show this, we have on applying U_1

$$W_\mu \to W_\mu'' = U_1 W_\mu U_1^{-1} + (\partial_\mu U_1) U_1^{-1}$$

Further applying U_2 , we get

$$\begin{aligned}W_\mu \to W_\mu'' &= U_2(U_1 W_\mu U_1^{-1} + (\partial_\mu U_1) U_1^{-1}) U_2^{-1} + (\partial_\mu U_2) U_2^{-1} \\ &= (U_2U_1) W_\mu (U_2U_1)^{-1} + \{U_2 \partial_\mu (U_1) + \partial_\mu (U_2) U_1\}(U_1^{-1}U_2^{-1}) \\ &= (U_2U_1) W_\mu (U_2U_1)^{-1} + \left(\partial_\mu (U_2U_1)\right)(U_2U_1)^{-1}\end{aligned} \tag{14}$$

The consistency of (13) requires that term $(\partial_\mu U)U^{-1}$ on the right-hand side should also satisfy Lie-algebracommutation rule. An infinitesimally small variation dx^μ will change U by a small amount because U(x) isa differentiable function of the space-time coordinates. Therefore

$$U(x^\mu + dx^\mu) = \left(I + c_\mu^a(x) dx t_a + \cdots\cdots\right) U(x) \tag{15}$$

Also using Taylor expansion for the left-hand side, we get

$$U(x^\mu + dx^\mu) = U(x) + \partial_\mu U(x) dx^\mu + \cdots\cdots \tag{16}$$

Therefore comparing dx^μterm in (15) and (16), we have

$$\left(\partial_\mu U(x)\right) U^{-1} = c_\mu^a t_a \tag{17}$$

which is being proportional to the generator t_a follows Lie-algebra. Further using Eq.(3) we have

$$\begin{aligned}\partial_\mu U(x) &= \partial_\mu \xi^a t_a \exp(\xi^b t_b) = \partial_\mu \xi^a t_a U \\ \text{Or} \quad \left(\partial_\mu U(x)\right) U^{-1} &= \partial_\mu \xi^a t_a\end{aligned} \tag{18}$$

The first term in Eq.(13) $UW_\mu U^{-1}$ follows lie algebra. This is because if a group element X, is sandwichedbetween the transformation matrix U and its inverse, is again an element ofthe group,

$$\text{i.e. if } X, U \in G, Y = UXU^{-1} \in G. \tag{19}$$

X, being member of G, we have

$X = I + \xi^a t_a + O(\xi^2)$,

Also $U = \exp(\xi^a t_a) \approx I + \xi^a t_a + O(\xi^2)$,

Let $Y = UXU^{-1} = I + \bar{\xi}^a t_a + O(\bar{\xi}^2)$,

Substituting this into Eq.(191919), We get

$$\begin{aligned}Y &= I + \bar{\xi}^a t_a + O(\bar{\xi}^2) \\ &= \left(I + \xi^a t_a + O(\xi^2)\right)(I + \xi^a t_a + O(\xi^2)) \\ &\quad (I - \xi^a t_a + O(\xi^2)) \\ &= I + \xi^a t_a + O(\xi^2)\end{aligned}$$

This exercise shows that $\bar{\xi}^a = O(\xi^a)$ .Therefore, $\bar{\xi}^a$ can also be expanded in a power series in ξ^a. Thereforethe term $UW_\mu U^{-1}$ in (13) is also Lie-algebra valued and is called an adjoint representation of W_μ. Therefore the conclusion is that the gauge fields W_μ transform in an adjoint representation of the group$(UW_\mu U^{-1})$, added with an inhomogeneous but Lie algebra valued term $(\partial_\mu U)U^{-1}$. It is easy to evaluate transformation rule for components W_μ^a by using Eq.(3), Eq.(12)and Eq.(13)

$$W_\mu^a \to (W_\mu^a)' = W_\mu^a + f_{bc}^a \xi^b W_\mu^c + \partial_\mu \xi^a + O(\xi^2) \tag{20}$$

This transformation law differs from that of the abelian gauge fields by thepresence of the term $f_{bc}^a \xi^b W_\mu^c$. Eq. (20) can also be written

$$\delta W_\mu^a = D_\mu \xi^a + O(\xi^2) \tag{21}$$

Where ξ^a is a quantity which transforms in the adjoint representation of the gauge group. Here D_μ is the generator of covariant translations, which are infinitesimal space-time translations and infinitesimal field-dependent gauge transformations preserving the covariant nature of translated quantity. This is equally true for infinitesimal abelian and non-abelian transformations.

The transformations satisfy $\delta(\varphi\psi) = (\delta\varphi)\psi+\varphi(\delta\psi)$, Therefore the Leibniz' rule is applicable to covariant derivatives i.e.

$$D_\mu(\varphi\psi) = (D_\mu\varphi)\psi + \varphi(D_\mu\psi) \tag{22}$$

The covariant derivatives depend on the representation of the fields on which they act through the choice of the generators t^a. Hence each of the three terms in Eq.(22) may contain a different representation for the generators.

Let us understand this from a simpler example for abelian transformations. Let ψ_1 and ψ_2 transform under local phase transformations with strength q_1 and q_2 respectively, then we have

$$\begin{aligned} D_\mu(\psi_1\psi_2) &= (\partial_\mu - i(q_1+q_2)A_\mu)(\psi_1\psi_2), \\ D_\mu\psi_1 &= (\partial_\mu - iq_1A_\mu)\psi_1, \\ D_\mu\psi_2 &= (\partial_\mu - iq_2A_\mu)\psi_2, \end{aligned} \tag{23}$$

With these definitions it is straight forward to verify Eq. (22).

However, unlike ordinary differentiations, two covariant differentiations for non abelian fields do not commute. The commutator of two covariant derivatives D_μ and D_ν for non abelian fields, which is obviously a covariant quantity, is given by

$$\begin{aligned} [D_\mu, D_\nu]\psi &= D_\mu(D_\nu\psi) - D_\nu(D_\mu\psi) \\ &= -(\partial_\mu W_\nu - \partial_\nu W_\mu - [W_\mu, W_\nu])\psi \end{aligned} \tag{24}$$

The quantity in parenthesis is a covariant anti-symmetric tensor $G_{\mu\nu}$,

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - [W_\mu, W_\nu] \tag{25}$$

which is called as the field strength. This quantity is analogous to abelian case field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It differs from the abelian field strength $F_{\mu\nu}$ by the presence of the term quadratic in W_μ. We saw by Eq. (13) that gauge field W_μ does not transform covariantly. The covariant object associated with the gauge field is just the field strength $G_{\mu\nu}$. Both $D_\nu D_\mu\psi$ and $D_\mu D_\nu\psi$ transform covariantly under the gauge transformations, therefore the field strength must transform covariantly according to

$$G_{\mu\nu} \to G'_{\mu\nu} = UG_{\mu\nu}U^{-1} \tag{26}$$

Further W_μ is Lie-algebra valued and the quadratic term in (25) is a commutator, therefore field strength $G_{\mu\nu}$ is also Lie-algebra valued and is decomposed in terms of the group generators t_a, mathematically

$$G_{\mu\nu} = G^a_{\mu\nu}t_a \tag{27}$$

where the matrices $G^a_{\mu\nu}$ is shown as

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - f^a_{bc}W^b_\mu W^c_\nu \tag{28}$$

Unlike abelian field strength $F_{\mu\nu}$, $G_{\mu\nu}$ is not invariant under an infinitesimal transformation, it transforms as

$$G_{\mu\nu} \to G'_{\mu\nu} = G_{\mu\nu} + [\xi^a t_a, G_{\mu\nu}] \tag{29}$$

Or in terms of components

$$G^a_{\mu\nu} = (G^a_{\mu\nu})' = G^a_{\mu\nu} + f^a_{bc}\xi^b G^c_{\mu\nu} \tag{30}$$

The adjoint representation is trivial for an abelian group, therefore the structure constants f^a_{bc} vanish for the abelian transformations and the field strengths are invariant.

Using Eq. (24), we can write

$$[D_\mu, D_\nu] = -G_{\mu\nu} \tag{31}$$

This result is known as the *Ricci identity*. This implies that the commutator of two covariant derivatives for a non-abelian field is equal to an infinitesimal gauge transformation with parameters $-G^a_{\mu\nu}$.

Further, as for the abelian case, using Eq.(31) and applying Jacobi identity, we get the cyclic combination of three covariant derivatives sum up to zero, that is

$$\begin{gathered} [D_\mu[D_\nu, D_\rho]] + [D_\nu[D_\rho, D_\mu]] + [D_\rho[D_\mu, D_\nu]] = 0 \\ \text{Or} \quad D_\mu G_{\nu\rho} + D_\nu G_{\rho\mu} + D_\rho G_{\mu\nu} = 0 \end{gathered} \tag{32}$$

Since $D_\mu \equiv \partial_\mu - W_\mu$, it can be shown that

$$D_\mu G_{\nu\rho} = \partial_\mu G_{\nu\rho} - [W_\mu, G_{\nu\rho}] \tag{33}$$

or, in components,

$$D_\mu G^a_{\nu\rho} = \partial_\mu G^a_{\nu\rho} - f^a_{bc}W^b_\mu G^c_{\nu\rho} \tag{34}$$

Eq.(32) is known as the Bianchi identity and in the abelian case it corresponds to the Homogeneous Maxwell equations. As we know the integral form of the homogeneous Maxwell equation implies the conservation of magnetic flux. The non-abelian analogue of the magnetic flux is not conserved and is carried away by the non-abelian gauge fields because of the presence of commutator term in Eq.(33) which is zero in the case of an abelian transformation. This suggests the existence of a source of magnetic lines of force i.e. a magnetic monopole.

III. RESCALING OF THE GAUGE FIELD AND COUPLING CONSTANT

The coupling constant, g, rescales the gauge fields $W_{a\mu}$ to $gW_{a\mu}$. If the gauge group is a product of different subgroups, then the gauge fields corresponding to each one of these subgroups is rescaled by an independent coupling constant. The number of coupling constants is thus equal to the number of independent gauge-invariant terms that depend exclusively on the gauge fields.

Let us consider a simple gauge group which is defined as a group without invariant subgroups. Therefore there is only one coupling constant. The quantities used in above discussion will require the following substitutions

$$W^a_\mu \to gW^a_\mu, \qquad G^a_{\mu\nu} \to gG^a_{\mu\nu}, \qquad \xi^a \to g\xi^a \tag{35}$$

$$D_\mu\psi \equiv \partial_\mu\psi - gW_\mu\psi \tag{36}$$

The Ricci identity (31) now modifies to

$$[D_\mu, D_\nu] = -gG_{\mu\nu} \tag{37}$$

and the field strength tensor

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - g[W_\mu, W_\nu] \tag{38}$$

The components $G^a_{\mu\nu}$ defined by the relation $G_{\mu\nu} = G^a_{\mu\nu} t_a$ rescale to

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g f^a_{bc} W^b_\mu W^c_\nu \quad (39)$$

The gauge-transformation parameters are also rescaled, and therefore gauge transformation U becomes

$$U = \exp(g\xi^a t_a) \quad (40)$$

Therefore the field components W^a_μ, $G^a_{\mu\nu}$ and the matter field ψ transform under infinitesimal gauge transformations as

$$\delta W^a_\mu = \partial_\mu \xi^a - g f^a_{bc} W^b_\mu \xi^c = D_\mu \xi^a \quad (41)$$

$$\delta G^a_{\mu\nu} = g f^a_{bc} \xi^b G^c_{\mu\nu} \quad (42)$$

$$\delta\psi = g\xi^a t_a \psi \quad (43)$$

The quantity $\xi_a$whilein rewriting the gauge field transformation as a covariant derivative is treated like a quantity that transforms in the adjointrepresentation of the group.

IV. GAUGE THEORY OF SU(2)

The gauge-invariant Lagrangians for spin-0 and spin-1/2 fields can now be constructed using covariant derivatives. Let us consider a set of N spinor fields $\psi_i$transforming under unitary transformations U i.e. $U^\dagger = U^{-1}$.This belongs to a certain group G,we have

$$\psi_i \to \psi'_i = U_{ij}\psi_j \quad (44)$$

(i, j = 1, . . . ,N). The indices i, j can be suppressed and ψ can be considered as an N-dimensional column vector

$$\psi \to \psi' = U\psi \quad (45)$$

Similarly we can write for $\bar{\psi}$

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi} U^\dagger \quad (46)$$

The massive Dirac Lagrangian,

$$\mathcal{L} = -\bar{\psi}_i \partial\!\!\!/ \psi_i - m\bar{\psi}_i\psi_i \quad (47)$$

This Lagrangian describesNspin- $\frac{1}{2}$(anti)particlesof mass m and is invariant under local G transformations.If the ordinary derivative in (47) is replaced bya covariant derivative (henceforth suppressing the indices i, j etc.)

$$\mathcal{L} = -\bar{\psi} D\!\!\!\!/\, \psi - m\bar{\psi}\psi$$
$$= -\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi + g\bar{\psi}\gamma^\mu W_\mu \psi, \quad (48)$$

Therefore the last term denotes interaction betweenψ and the gauge field$W_\mu (i.e. W^a_\mu t_a)$. This term is also known as the interaction Lagrangian, can berewritten as

$$\mathcal{L}_{int} = g W^a_\mu \bar{\psi}\gamma^\mu t_a \psi \quad (49)$$

The gauge transformations are unitary and t_a are the anti-Hermitean generator matrices which are the parameters of the representation gauge group G. The matrices $(t_a)_{ij}$arethe non-abelian charges and are proportional to abelian charges $F^{ij}_a(0)$up to a proportionality factor i. The commutationrelation for these charges is Lie algebra relation Eq.(5) which is necessary and sufficient condition required for the non-abelian gauge transformations to constitute a group.

Now let us construct a gaugeinvariant Lagrangian for fermions transforming as doublets under the group SU(2)which, as we know, consists of all 2×2 unitary matrices with unit determinant. These matrices are expressible as an exponential

$$U(\xi) = exp(g\xi^a t_a) \quad (a = 1,2,3) \quad (50)$$

Here ξ is the rotation or translational parameter in isospinspace.The coupling constant g is in accordance with Eq.(40).The three generators of SU(2) are expressed in terms of the isotopic spinmatricesσ_a

$$t_a = \frac{1}{2} i\tau_a, \quad (51)$$

whichare the Pauli matrices used in the context of ordinary spin,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (52)$$

It is easy to see that the generators $t_a$satisfy the commutationrelations

$$[t_a, t_b] = -\varepsilon_{abc} t_c, \quad (53)$$

ensuring that the transformation matrices in Eq.(50) form a group.

Historically an interesting early attempt to construct the covariant derivative on ψ was presented by O. Klein (1938) at the motivation of Yukawa's conjecture of the existence of thepion. It was based on a somewhat ad-hoc, extension ofgravity in five space-time dimensions, and led to theories which have features very similarto those of SU(2) and SU(2) × U(1) gauge theories. But the present day non-abelian gauge field theory was constructed by Yang and Mills and independently by Shaw [2]. It isbased on above discussed SU(2) matrices and is motivated by the existence of the approximate iso-spin invariance in Nature.The observation that the proton and neutron are almost degenerate in mass, and play an identical role in strong interaction processes,theproton and the neutron are regarded as anisospin(or isobaric spin) doublet. This doublet field,analogous to the $s = \frac{1}{2}$doublet of ordinary spin, is expressible as a column vector

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \quad (54)$$

The transformation from one isospin state to other is represented through the transformation matrix U which is related to transformation ξ and SU(2) group generators t_a through Eq.(50). The transformation of the state ψ to ψ' can be represented as

$$\psi \to \psi' = U\psi, \quad (55)$$

It is a matter of convention which component of ψcorresponds to the proton and which one tothe neutron because of the almost exact symmetry of isospin invariance.

The covariant derivative for transformation Eq.(55) is straight forward, using Eq. (36), (51), (52) and (54), we get

$$D_\mu\psi = \begin{pmatrix} \partial_\mu\psi_p \\ \partial_\mu\psi_n \end{pmatrix} - \frac{1}{2} ig W^a_\mu \tau_a \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}.$$
$$= \begin{pmatrix} \partial_\mu\psi_p \\ \partial_\mu\psi_n \end{pmatrix} - \frac{1}{2} ig \begin{pmatrix} W^3_\mu & W^1_\mu - iW^2_\mu \\ W^1_\mu + iW^2_\mu & -W^3_\mu \end{pmatrix} \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}.$$
$$= \begin{pmatrix} \partial_\mu\psi_p - \frac{1}{2} ig W^3_\mu \psi_p - \frac{1}{2} ig (W^1_\mu - iW^2_\mu)\psi_n \\ \partial_\mu\psi_n + \frac{1}{2} ig W^3_\mu \psi_n - \frac{1}{2} ig (W^1_\mu + iW^2_\mu)\psi_p \end{pmatrix} \quad (56)$$

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The ordinaryderivative is replaced by a covariant one in the Lagrangian of a degenerate doubletof spin$-\frac{1}{2}$fields. We have, using Eq. (48), (49) and (51),

$$\begin{aligned}\mathcal{L} &= -\bar{\psi}\not{D}\psi - m\bar{\psi}\psi \\ &= -\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi + \frac{1}{2}ig\bar{\psi}\, W_\mu^a \gamma^\mu \tau_a \psi,\end{aligned} \tag{57}$$

This forms what is known as a locally SU(2) invariant Lagrangian.**Note:** Dirac γ^μ matrices are as below

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \tag{58}$$

Rewriting Eq.(39), we have

$$G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g f_{bc}^a W_\mu^b W_\nu^c$$

Comparing Eq.(6) i.e. $[t_a, t_b] = f_{ab}^c t_c$ with Eq.(53), we get $f_{ab}^c = -\varepsilon_{abc}$, Therefore for SU(2) transformations, the field strength tensor is given by

$$G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\varepsilon_{abc} W_\mu^b W_\nu^c \tag{59}$$

Using Eq.(20), we can show under infinitesimal SU(2) transformations the gauge field W_μ^a transformsas

$$W_\mu^a \to \left(W_\mu^a\right)' = W_\mu^a + g\varepsilon_{abc} W_\mu^b \xi^c + \partial_\mu \xi^a \tag{60}$$

Weak interactions are governed by theories based on SU(2) gauge transformations. Butthe strong forces are governed by SU(3) transformations.Since we use color as the degrees of freedom transforming under SU(3) (like isospin in SU(2)), theSU(3) gauge theory is known as quantum chromodynamicsor QCD in short. The corresponding gauge fields in QCD are the gluon fields. The gluons are supposed to bind the quarks in hadrons.The quarks, like SU(2) doublet of proton and neutron (Eq.(54)) transform as triplets in SU(3) theory.

V. LAGRANGIAN FOR SPIN -0 FIELDS

The similar procedure is adopted for determining Gauge invariant Lagrangiansfor spin-0 fields.An array of complex scalar fieldsφ can be considered as a column vector. Let it transforms under transformationsU.

$$\varphi \to \varphi' = U\varphi \tag{61}$$

Then the complex conjugate field φ^* is a row vector,

$$\varphi^* \to (\varphi^*)' = \varphi^* U^\dagger \tag{62}$$

Using the equation for the covariant derivative i.e.$D_\mu \equiv \partial_\mu - gW_\mu$ andalsothe equation for decomposition of the gauge field i.e.$W_\mu = W_\mu^a t_a$, we can write

$$\begin{aligned} D_\mu\varphi &\equiv \partial_\mu\varphi - gW_\mu^a t_a \varphi \\ D_\mu\varphi^* &\equiv \partial_\mu\varphi^* - g\varphi^* t_a^\dagger W_\mu^a \end{aligned} \tag{63}$$

The transformation matrices U in (61) are unitary (so that $t_a^\dagger = -t_a$). Therefore it can be shown using the inner product $|\phi|^2 = \phi_i^*\phi_i$ that the following Lagrangian is gauge invariant,

$$\mathcal{L} = -\left|D_\mu\phi\right|^2 - m^2|\phi|^2 - \lambda|\phi|^4 \tag{64}$$

Substitution of Eq.(63) in Eq.(64), we get

$$\begin{aligned}\mathcal{L} = &-(\partial^\mu\phi^* + g\varphi^* t_a W^{a\mu})\left(\partial_\mu\phi - gW_\mu^b t_b\phi\right) - m^2|\phi|^2 \\ &- \lambda|\phi|^4\end{aligned}$$

$$\begin{aligned}\mathcal{L} = -\left|\partial_\mu\phi\right|^2 - m^2|\phi|^2 - \lambda|\phi|^4 \\ - gW^{a\mu}\left(\phi^* t_a\,\partial_\mu\phi - \left(\partial_\mu\phi^*\right)t_a\phi\right) \\ + g^2 W^{a\mu} W_\mu^b\left(\phi^* t_a t_b\phi\right)\end{aligned}$$

This showsthe generators $t_a$are matrix generalizations of the charge. It can be easily shown using $t_a = \frac{1}{2}i\tau_a$ and$\tau_a\tau_b + \tau_b\tau_a = 2\,I\,\delta_{ab}$ that Lagrangian is invariantunderSU(2).

$$\begin{aligned}\mathcal{L} = -\left|\partial_\mu\phi\right|^2 - m^2|\phi|^2 - \lambda|\phi|^4 - \frac{1}{2}igW^{a\mu}\left(\phi^*\tau_a\,\overleftrightarrow{\partial_\mu}\phi\right) \\ - \frac{1}{4}g^2\left(W_\mu^a\right)^2|\phi|^2\end{aligned} \tag{65}$$

VI. THE GAUGE FIELD LAGRANGIAN

The locally invariant Lagrangians constructed so far in the preceding section arefor matter fields. The procedure is same that the covariant derivative replaces theordinary one. The gauge fields are considered responsible for interaction between the matter fields for example fermions, quarks etc. They have not yet been treatedas new dynamical degrees of freedom. They themselves are the fields and so can produce action and a force. That means they also have a Lagrangian, and therefore an equation of motion. This Lagrangian should also be locally gaugeinvariant. This construction requires the use of the Lie algebra valued field strength tensor $G_{\mu\nu}$. We have from Eq.(26)

$$G_{\mu\nu} \to G_{\mu\nu}' = UG_{\mu\nu}U^{-1}$$

The product of these tensors can be written as

$$G_{\mu\nu}G_{\rho\sigma}\cdots G_{\lambda\tau} \to G_{\mu\nu}'G_{\rho\sigma}'\cdots G_{\lambda\tau}' = U\left(G_{\mu\nu}G_{\rho\sigma}\cdots G_{\lambda\tau}\right)U^{-1} \tag{66}$$

Consequently the trace of these arbitrary products is gauge invariant, i.e.

$$\mathrm{Tr}\left(G_{\mu\nu}G_{\rho\sigma}\cdots G_{\lambda\tau}\right) \to \mathrm{Tr}\left(UG_{\mu\nu}G_{\rho\sigma}\cdots G_{\lambda\tau}U^{-1}\right) = \mathrm{Tr}\left(G_{\mu\nu}G_{\rho\sigma}\cdots G_{\lambda\tau}\right) \tag{67}$$

by virtue of the cyclicity of the trace operation.

Therefore $Tr\left(G^{\mu\nu}G_{\mu\nu}\right)$ is the simplest Lorentz invariant and parity conserving term and

$$\mathcal{L}_G = \frac{1}{4g^2}Tr\left(G^{\mu\nu}G_{\mu\nu}\right) \tag{67}$$

is the simplest possible Lagrangian for Gauge fields. The factor$\left(\frac{1}{4g^2}\right)$ is used for normalisation of the Lagrangian.The other two simpler alternative Lorentz invariant forms G^{μ}_{μ} and $G_{\mu\nu}G_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}$,are not used in the construction of the lagrangian. This is because the first one vanishesdue to the anti-symmetry of$G_{\mu\nu}$. The form $G_{\mu\nu}G_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}$ does not conserve parity. It can be shown that it is actually equal to the total divergenceof Chern–Simons term $\varepsilon^{\mu\nu\rho\sigma}\,Tr\left(W_\nu\partial_\rho W_\sigma - \frac{2}{3}gW_\nu W_\rho W_\sigma\right)$i.e.

$$\varepsilon^{\mu\nu\rho\sigma}\,Tr\left(G_{\mu\nu}G_{\rho\sigma}\right) = 4\,\partial_\mu\left\{\varepsilon^{\mu\nu\rho\sigma}\,Tr\left(W_\nu\partial_\rho W_\sigma - \frac{2}{3}gW_\nu W_\rho W_\sigma\right)\right\} \qquad (68)$$

and so it gets ignored while evaluating the action, at least in perturbation theory but not when one is dealing with non-trivial boundary conditions.

Using Equation (39) and rescaling W_μ *to* gW_μ, we get the Lagrangian takes the form

$$L_W = Tr\,(t_a t_b)\left\{\frac{1}{4}\left(\partial\mu W_\nu^a - \partial_\nu W_\mu^a\right)\left(\partial_\mu W_\nu^b - \partial_\nu W_\mu^b\right) - g f^a_{cd} W_\mu^c W_\nu^d \partial_\mu W_\nu^b + \frac{1}{4}g^2 f^a_{cd} f^b_{ef} W_\mu^c W_\mu^e W_\nu^d W_\nu^f\right\} \qquad (69)$$

Therefore the Gauge field Lagrangian consists of a kinetic term, which is similar to one in the abelian Lagrangian, a$(W)^3$-interaction terms and a $(W)^4$-interaction terms which depend on structure constants of the gauge group.

There is a tensor that multiplies the terms in aboveequation given as

$$g^R_{ab} = Tr\,(t_a t_b) \qquad (70)$$

Superscript R indicates that it depends on the representation taken for the generators t_a. The generators in the adjoint representation definewhat is known asCartan-Killing metric,

$$g_{ab} = f^c_{ad} f^d_{bc} \qquad (71)$$

These two representation equations define a totally anti-symmetric tensor in all indices

$$f^R_{abc} \equiv -f^d_{ab}\,g^R_{dc} \qquad (72)$$

This is because using (70), we can write

$$f^R_{abc} = -f^d_{ab}Tr\,(t_d t_c) = -Tr\,(t_a t_b t_c - t_b t_a t_c) \qquad (73)$$

The right hand side of this equation is anti-symmetric under the exchange of any two indices by virtue of cyclicity of the trace.

It is easy to show that, g^R_{ab} constitutes a groupinvarianttensorfor any representation. Arank-2 tensor T_{ab} inanadjoint representationT_{ab} transformsas in Eq. (30)

$$T_{ab} \to T_{ab} - \xi^c\left(f^d_{ca}T_{db} + f^d_{cb}T_{ad}\right) + O(\xi^2) \qquad (74)$$

Here asg^R_{ab} is a symmetric tensor as $Tr(t_a t_b) = Tr(t_b t_a)$. Let us substitute$g^R_{ab}$ forT_{ab}, we obtain

$$g^R_{ab} \to g^R_{ab} - \xi^c\left(f^R_{cab} + f^R_{cba}\right) + O(\xi^2)$$

Or $$\delta g^R_{ab} \to -\xi^c\left(f^R_{cab} + f^R_{cba}\right) + O(\xi^2) \qquad (75)$$

But δg^R_{ab} vanishes because of the total anti-symmetry of f^R_{abc}. This proves the invariance of the tensors g^R_{ab}.

The complex conjugate matricest^*_a also satisfy the Lie-algebra Eq.(5) with real structure constants. Therefore the complex conjugate matrices t^*_a also generate a group representation. Therefore, ifg^R_{ab} is complex, both its real and imaginary parts will separately constitute an invariant tensor.

The invariant tensors g^R_{ab} depend on the representation, asfor an abelian group, Cartan-Killing metric (71) vanishes trivially as the structure constants are zero. The abelian groups have nontrivial representations for which (70) does not vanish. The invariant tensors g^R_{ab} can be shown equal to proportionality constant for the simple groups.

The tensor g^R_{ab} or at least its real or imaginary part for any arbitrary representation can be diagonalisedwith eigenvalues equal to 1,−1 or 0. This is achieved by suitably redefining the gauge fields and therefore, the group parameters$\xi^a$and the corresponding structure constants.This is becauseg^R_{ab} has only nonzero eigenvalues of equal sign that the first term in theLagrangian (69) is kinetic term for each of the gauge fields of thesame normalization. The tensor g^R_{ab} has non-positive eigenvalues. This is because for the so-called compact Lie groups with property$t^\dagger_a = -t_a$.Further $Tr(t_a t_b)$is replaced by $-\delta_{ab}$ in the Lagrangianas the lie groups are compact.Therefore from Eq.(69) we get,

$$L_W = -\frac{1}{4}\left(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a\right)^2 + g f_{abc} W^{a\mu} W^{b\nu} \partial_\mu W_\nu^c - \frac{1}{4}g^2 f_{abc} f_{ade} W^{b\mu} W_\mu^d W^{c\nu} W_\nu^e \qquad (76)$$

The gauge-field Lagrangians for non-compact groups, for example for Lorentz group in the case of theory of Gravity,is evaluated by evaluatingthe Feynman diagrams for g^R_{ab} having non- zero eigen-values.However thegauge fields in the theory of Gravity have rather different roles.

Adding the gauge-field Lagrangian (76) with a gauge invariant Lagrangian for the matter fields, for example for fermions in Eq.(48), we can write

$$\mathcal{L} = \mathcal{L}_W + \mathcal{L}_\psi = \frac{1}{4}Tr\,(G_{\mu\nu}G^{\mu\nu}) - \bar{\psi}\not{D}\psi - m\bar{\psi}\psi \qquad (77)$$

The field equations are the corresponding Euler-Lagrangeequations evaluated using Hamilton's principle These equations for fermions are covariant version of the Dirac equation, i.e.,

$$(\not{D} + m)\psi = 0,\ \ \bar{\psi}(\not{D} - m) = 0, \qquad (78)$$

or using $t^\dagger_a = -t_a$, we can write

$$(\not{\partial} + m)\psi = gW^a t_a \gamma^\mu \psi\,,$$
$$\bar{\psi}\left(\overleftarrow{\not{\partial}} + m\right) = -g\bar{\psi}\gamma^\mu t_a W_\mu^a\,, \qquad (79)$$

These equations are the non-abelian generalizations of abelian field equations.

The derivation of field equations for the gauge fields is more complicated. Let us consider an infinitesimal transformation causing a small variation in the field W_μ

$$W_\mu \to W_\mu + \delta W_\mu \qquad (80)$$

Here δW_μ is a covariant quantity transforming in the adjointrepresentation.

The corresponding change in field strength is

$$G_{\mu\nu} \rightarrow G_{\mu\nu} + D_\mu(\delta W_\nu) - D_\nu(\delta W_\mu), \quad (81)$$

Where covariant derivative on δW_ν is defined as

$$D_\mu(\delta W_\nu) = \partial_\mu(\delta W_\nu) - g\,[W_\mu, \delta W_\nu] \quad (82)$$

As $G_{\mu\nu}$ is anti-symmetric in μ and ν, the variation in action due to this transformation can be shown as

$$\delta\left(\frac{1}{4}\int d^4x Tr\left(G^{\mu\nu} G_{\mu\nu}\right)\right) = \int d^4x Tr\left(G^{\mu\nu} D_\mu(\delta W_\nu)\right) \quad (83)$$

The integrand on the right-hand side can be written as

$$D_\mu\{Tr(G^{\mu\nu}\delta W_\nu)\} - Tr\left(D_\mu G^{\mu\nu}(\delta W_\nu)\right) \quad (84)$$

$Tr(G^{\mu\nu}\delta W_\nu)$ is gauge invariant as δW_μ is a covariant object. Therefore replacing covariant derivative by an ordinary derivative and using Gauss' theorem, the first term converts into a surface integral over the boundary of the integration domain in (83). The variation δW_μ however,vanishes at the boundariesin accordance with Hamilton's principle. Therefore, only the second term in (84) contributes. Therefore from Eq.(83) , we have

$$\delta\left(\int d^4x\mathcal{L}_w\right) = -\int d^4xTr\left((D^\mu G_{\mu\nu})\delta W^\nu\right) \quad (85)$$

Similarly the variation of the matter action due to (80) is given as

$$\delta\left(\int d^4x\mathcal{L}_\psi\right) = -\int d^4xTr(\delta W^\mu J_\mu) \quad (86)$$

Changing dummy index from μ to ν and adding Eq.(86) toEq.(85), we get

$$\delta\int d^4x\,(\mathcal{L}_w + \mathcal{L}_\psi) = -\int d^4xTr\left((D^\mu G_{\mu\nu} + J_\nu)\delta W^\nu\right)$$

The variation in Lagrangian should be zero, therefore the field equation for the gauge field are

$$D_\mu G_{\mu\nu} + J_\nu = 0\,, \quad (87)$$

Both $G_{\mu\nu}$ and J_μ are Lie-algebra valued matrices. Using $Tr(t_a t_b) = -\delta_{ab}$ and $J_\mu = J_\mu^a t_a$ the variation of the gauge field and the matter action can be written as

$$\delta\left(\int d^4x\mathcal{L}_w\right) = \int d^4x\delta W^{a\nu} D^\mu G^a_{\mu\nu} \quad (88)$$

$$\delta\left(\int d^4x\mathcal{L}_\psi\right) = \int d^4x\delta W^{\mu a} J^a_\mu \quad (89)$$

Therefore, in terms of components, we get

$$D^\nu G^a_{\mu\nu} = J^a_\mu \quad (90)$$

Writing in terms of Gauge field W, we get a rather complicated equation

$$\partial^\nu\left(\partial\mu W^a_\nu - \partial_\nu W^a_\mu\right) + gf^a_{bc}\left(W^{b\nu}\partial_\mu W^c_\nu + W^b_\mu \partial_\nu W^{c\nu} - 2W^{b\nu}\partial_\nu W^c_\mu\right) + g^2 f^a_{bc} f^c_{de} W^d_\mu W^b_\nu W^{e\nu} = J^a_\mu \quad (91)$$

The current J^a_μ can be evaluated as

$$J^a_\mu = g\bar{\psi}\gamma_\mu t_a\psi \quad (92)$$

Applying a covariant derivative D_μ and raising indices by two in Eq.(90), we get

$$D_\mu D_\nu G^{a\mu\nu} = D^\mu J^a_\mu \quad (93)$$

On left hand side using antisymmetry of $G_{a\mu\nu}$ and Ricci identity (27), we get

$$D_\mu D_\nu G^{a\mu\nu} = \frac{1}{2}[D_\mu, D_\nu]G^{a\mu\nu} \quad (94)$$

$$= -\frac{1}{2}gG^b_{\mu\nu} f^a_{bc} G^{c\mu\nu}$$

Right hand side vanishes because of anti-symmetricnature of f^a_{bc}. Therefore from Eq.(93), we get

$$D^\mu J^a_\mu = 0 \quad (95)$$

Or raising indices by two and rescaling W to gW, we have from Eq.(34) that

$$D^\mu = \partial^\mu - gf^a_{bc} W^b_\mu$$

Therefore from Eq.(95), we get

$$\partial^\mu J^a_\mu - gf^a_{bc} W^{b\mu} J^c_\mu = 0 \quad (96)$$

The above result in Eq. (95) shows that the gaugefields couple tocurrents thatare covariantly constant. However the charges associated with the current are not quite conserved as evident from Eq.(96). This is obviously contribution of non-neutral gauge fields must be included to define charges for their conservation.

VII. CHIRAL GAUGE GROUPS AND DISCRETE SYMMETRIES

There is lot of literature available on topics such as discrete symmetries, like parity reversal and charge conjugation, also on topics like chiral spinorfields, Eigenspinors of the γ_5 matrix. We skip our discussion on these terms.

Let us consider non-abelian group of gauge transformations based on generators t_a that give rise to fermionic vector or axial-vector currents. The axial vectors are those quantities which do not change direction under mirror reflection such as angular velocity of rotating wheel of a receding car on road. More on axial vectors can be seen in the literature available on internet.

Further the matrices $t_a\gamma_5$ act on both the gauge-group and the spinor indicesbutdo not generate a group of transformations.This is because the group of matrices $t_a\gamma_5$ is not closed under commutation

$$[t_a\gamma_5, t_b\gamma_5] = f^c_{ab} t_c \neq f^c_{ab} t_c\gamma_5 \quad (97)$$

The dimension of the group and thus the number of independent gauge field doubles if both t_a and $t_a\gamma_5$ are taken as independent generators.Therefore chiral projection operators defined by $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ are used. These Projectionoperators satisfy $P^2_\pm = P_\pm$, and therefore the matrices $t_a P_\pm$ do generate a representation group. We can verify the matrices $t_a P_\pm$ satisfy lie-algebra i.e.

$$[(t_a P_\pm), (t_b P_\pm)] = f^c_{ab}(t_c P_\pm) \quad (98)$$

The fermion fieldsdecompose into chiral components $\psi_L = P_+\psi$ and $\psi_R = P_-\psi$. These chiral components ψ_L and ψ_R form independent representations of the gauge group.Also t^R_a and t^L_a, which are a priori unrelated but satisfythe same commutation relations as given by Eq.(98).These two sets of generators of the gauge group act on independent arrays of fields denoted by ψ_R and ψ_L, respectively. Thecoupling of the gauge fields to the fermions thereforeis represented in terms of the chiralprojectors $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ and is proportional to $\gamma_\mu(1+\gamma_5)t^L_a + \gamma_\mu(1+\gamma_5)t^R_a$.

There are a number of possibilities.Firstly$t_a^{\mathrm{R}}$and t_a^{L}arein-equivalent. Then the numbers of left- andright-handed fields is not equal. Second possibility is that only one oft_a^{R}or $t_a^{\mathrm{L}}$is present, then the gauge fields coupleto a vector or an axial vector, i.e. eitherto$\gamma_\mu(1+\gamma_5)$or to $\gamma_\mu(1-\gamma_5)$. That means the only field present is of one chirality. The third possibility is that $\psi_L$and$\psi_R$transformequivalently, or $t_a^{\mathrm{L}} = t_a^{\mathrm{R}}$. This always leads to a purely vector-like coupling. The fourth possibility is that the generators are each other's complexconjugates $t_a^{\mathrm{L}} = (t_a^{\mathrm{R}})^*$. Acomplex conjugate representation is always possible as the complexconjugate generators satisfy the same commutator algebra

$$[(t_a)^*,\ (t_b)^*] = f_{ab}^c(t_c)^* \tag{99}$$

The representation is real so it is possible that all generators are imaginary. An argument similar to Eq.(97) shows that this is possible only if the group is abelian.Therefore the gauge fieldsassociated with real generators have vectorlike and those with imaginary generators haveaxial-vectorlike coupling to fermions.

The generators for charge-conjugated fields ψ_L^c and ψ_R^c, are the complex conjugate matrices $(t_a^{\mathrm{R}})^*$ and $(t_a^{\mathrm{L}})^*$ respectively. Thereforeif the left-handed fields, $\psi_L$and ψ_L^c form a basisof independent fermion fields, the reducible representation with generators is given as

$$\begin{pmatrix} t_a^{\mathrm{L}} & 0 \\ 0 & \left(t_a^{\mathrm{R}}\right)^* \end{pmatrix} \tag{100}$$

These matrices also satisfy therelevant commutation relations associated with the gauge group. Similar representation exists for the right-handed fields$\psi_R$and $\psi^c{}_R$. However, the gauge transformations act identically on right-andleft-handed components and the representation is called as vector-like. This is because thecorresponding fermionic currents contain no axial vector like term $\gamma_\mu\gamma_5$ terms because the generators in Eq.(100) can bewritten in real form by an appropriate change of basis. Therefore the representationbased on these generators is real. Further it is possible to write a standard gauge-invariant mass term for a real representation which involves the products of right- and left-handed spinors. Thevectors like representationsarealso free of so-called anomalies.

The invariance of gauge theory under discretetransformations such as P, C or the combined transformation CP, requires that the covariantderivatives of the spinors transform just as ordinary derivatives underthe transformations. The covariant derivatives of ψ_L and $\psi_R$containmatrices $W_\mu^{\mathrm{L}} = W_\mu^a t_a^L$and $W_\mu^{\mathrm{R}} = W_\mu^a t_a^R$and read

$$D_\mu\psi_L = \partial_\mu\psi_L - W_\mu^L\psi_L,\ \ D_\mu\psi_R = \partial_\mu\psi_R - W_\mu^R\psi_R \tag{101}$$

Therefore transformationrules for the gauge fields can be determined.

Let us start with parity reversal, under which left- and righthandedspinors are interchanged.The gauge fields under parity reversal must transform as,

$$W_\mu^a(\boldsymbol{x}, x^0)\ t_a^L \xrightarrow{P} -(1\ -2\delta_{\mu 0})W_\mu^a(-\boldsymbol{x}, x^0)\ t_a^R \tag{102}$$

and vice versa for the covariant derivatives of $\psi_L$and$\psi_R$to transform into each other. The action of P on the gauge fields is fixed, and adding an extra phase factor to it is not possibleif the gauge-group representation is once chosen. However, Eq. (102) requires that the generators t_a^L and t_a^R are linearly dependent. Therefore there are twosituations.The representations are equivalent or each other's complex conjugate. The gauge fieldstransform as vectors under parity reversal and the theory is vectorlike in the first case. The gauge fields, in the second case, associated with the real generators transform asvectors while gauge fields associated with imaginary generators transform asaxial-vectors underparity reversal.

Thecovariant derivatives under charge conjugation, are given by the expression as below

$$D_\mu\bar{\psi}_L^T = \partial_\mu\bar{\psi}_L^T - \bar{W}_\mu^L\bar{\psi}_L^T$$

Or

$$\bar{\psi}_L\overleftarrow{D_\mu} = \bar{\psi}_L\left(\overleftarrow{\partial_\mu} - W_\mu^\dagger\right)$$

The gauge field must transform under charge conjugationas below to obtain the correct transformation of the covariant derivative

$$W_\mu^a(x, x^0)t_a^L \xrightarrow{C} W_\mu^a(x, x^0)(t_a^R)^* \tag{103}$$

This result is meaningful for (i) equivalent left- and right-handedrepresentations and (ii)complex conjugate left- and right-handed representations. In the first case the gauge fieldsassociated with real generators are even and gauge fields with purely imaginarygenerators are odd under charge conjugation. In the second case all fieldsare even.

Theparityand charge conjugation can be separately defined if spinors of bothchirality are present.However this is not so for CP conjugation. Combining the previous results Eq.(102) and Eq.(103) theCPtransformation of the gauge fields is given by

$$W_\mu^a(x, x^0)t_a \xrightarrow{CP} -(1\ -2\delta_{\mu 0})W_\mu^a(-x, x^0)(t_a)^* \tag{104}$$

Again there is no possibility for assigning an arbitrary phase factor. The invariance under CP requires the relation (104) to hold good for all irreducible representations.The CP phasefactor clearly depends on whether the gauge field has a real or purelyimaginary generator.

CONCLUSIONS

The non-abelian transformations produce different outcomes if the order of transformations is changed. The generators of the representation group of non-abelian transformations follow Lie algebra $[t_a, t_b] = f_{ab}^c t_c$ and are therefore non-commuting.Non-abelian transformations in isospin space SU(2) successfully generates Lagrangian for spin -1/2, spin 0 and also for Gauge fields, which are considered responsible for interaction between the matter fields. Non-abelian transformations in SU(3) space known as chromodynamics explain gluon interactions between the Quark fields. Discrete symmetries like parity and charge conjugation are also explained by the non-abelian transformations

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