A Study on Graph Theory of Path Graphs

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ABSTRACT : A simple graph G = (V, E) consists of V , a nonempty set of vertices, and E, a set of unordered pairs of distinct elements of V called edges. Simple graphs have their limits in modeling the real world. Instead, we use multigraphs, which consist of vertices and undirected edges between these vertices, with multiple edges between pairs of vertices allowed. In the mathematical field of graph theory, a path graph or linear graph is a graph whose vertices can be listed in the order v1, v2, …, vn such that the edges are {vi, vi+1} where i = 1, 2, …, n − 1. Equivalently, a path with at least two vertices is connected and has two terminal vertices (vertices that have degree 1), while all others (if any) have degree 2.

Paths are often important in their role as subgraphs of other graphs, in which case they are called paths in that graph. A path is a particularly simple example of a tree, and in fact the paths are exactly the trees in which no vertex has degree 3 or more. A disjoint union of paths is called a linear forest.

Key words: *graph theory of path graphs*

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INTRODUCTION

This work deals with graph labeling. All the graphs considered here are finite and undirected. The terms not defined here are used in the sense of Harary.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

By a (p, q) graph G, we mean a graph G = (V, E) with |V| =p and |E|=q.

Graph labeling was first introduced in the late 1960's. Many studies in graph labeling refer to Rosa's research in 1967. Rosa introduced a function f from a set of vertices in a graph G to the set of integers {0, 1, 2,..., q} so that each edge xy is assigned the label |f(x) - f (y)|, all labels are distinct. Rosa called this labeling β-valuation. Independently, Golomb studied the same type of labeling and called as graceful labeling.

Graceful labeling originated as a means of attacking the conjecture of Ringel that K2n+1 can be decomposed into 2n +1 subgraphs that are all isomorphic to a given tree with n edges. Those graphs that have some sort of regularity of structure are said to be graceful. Sheppard has shown that there are exactly q! gracefully labeled graphs with q edges. Rosa suggested three possible reasons why a graph fails to be graceful.

1) *G* has "too many vertices" and "not enoughedges"
2) *G* has "too manyedges"
3) *G* has "wrongparity"

Labeled graphs serve as useful models for a broad range of applications such as X-ray crystallography, radar, coding theory, astronomy, circuit design and communication network addressing. Particularly, interesting applications of graph labeling can be found in [1, 2, 3, 4, 5, 9, 20].

In the recent years, dozens of graph labeling techniques such as a-labeling, k and (k, d) -graceful, k-equitable, skolem graceful, odd graceful and graceful like labeling have been studied in over 1000 papers [7].

In 1990, Antimagic graphs was introduced by Hartsfield and Ringel [11]. A graph with q edges is said to be antimagic if its edges can be labeled with 1, 2, ..., q such that the sums of the labels of the edges incident to each vertex are distinct. This notion of antimagic labeling was extended to hypergraphs by Sonntag [24].

In 1985, Lo [17] introduced the edge - graceful graph which is a dual notion of graceful labeling. A graph G(V,E) is said to be edge — graceful if there exists a bijection f from E to {1, 2, ..., q } such that the induced mapping f+ from V to {0, 1, 2,

..., p —1} given by f + (x) =(Σf (xy)) (mod p) taken over all edges xy is a bijection.

Every edge - graceful graph is found to be antimagic and Lo [17] found a necessary condition for a graph with p vertices and q edges to be edge -graceful asq (q = 1) $\frac{P(P+1)}{2}(mod\ p)$.

Lee [14] noted that this necessary condition extends to any multigraph with pvertices and q edges. He conjectured that any connected simple (p,q)graph with (mod p) q (q = 1) $\frac{P(P+1)}{2}(mod\ p)$. is edge - graceful and found that this condition issufficient for the edge - gracefulness of connected graphs. He also [15] conjectured thatall trees of odd order are edge - graceful.

Small [23] has proved that spiders in which every vertex has odd degree with the property that the distance from the vertex of degree greater than 2 to each end vertex is the same, are edge - graceful. Keene and Sirnoson [12] proved that all spiders are edge - graceful if the spider has odd order with exactly three end vertices. Cabaniss, Low, and Mitchern [6] have shown that regular spiders of odd order are edge - graceful. Lee

Seah, and Wang [16] gave a complete characterization of edge - graceful p k n Graphs
Shiu, Lam, and Cheng [22] proved that the composition of the path P3 and any null graph of odd order is edge - graceful.

Lo proved that all odd cycles are edge - graceful. Wilson and Riskin [25] proved that the Cartesian product of any number of odd cycles is edge - graceful.

This thesis emerged with the growing interest in the notion of edge -graceful graphs and its noteworthy conjectures.
By the necessary condition of edge - gracefulness of a graph one can verify that even cycles, and paths of even length are not edge - graceful. But whether trees of odd order are edge - graceful is still open.
Motivated by the notion of edge - graceful graphs and Lo's conjecture, we define a new type of labeling called strong edge - graceful labeling by relaxing its range through which we can get strong edge – graceful labeling of even order trees.

A (p,q) graph G is said to have strong edge – graceful labeling if there exists an injection f from the edge set to 2 0 13 so that theinduced mapping f+ from the vertex set to {0,1, ..., 2p –1} defined byf+(x)= Σ{f(xy)}|xyε E(G)}(mod2p)aredistinct.AgraphGissaidtobestrong edge – graceful if it admits a strong edge – gracefullabeling.

In this thesis, we investigate strong edge – graceful labeling (SELL) of some graphs. In some situations, we also present edge – graceful labeling of some trees towards attempting to the Lo's Conjecture.

PRELIMINARIES

Definition 2.1
A graph G = (V, E) is a finite non-empty set V of objects called vertices together with a set E of unordered pairs of distinct vertices called edges.
Definition 2.2
The cardinality of the vertex set of graph G is called the order of G and is denoted by p. The cardinality of its edge set is

called the size of G and is denoted by q. A graph with p vertices and q edges is called a (p, q) — graph.

Definition 2.3

If e = (u, v) is an edge of G, we write e = uv and we say that u and v are adjacent vertices in G. If two vertices are adjacent, then they are said to be neighbours. Further, vertex u and edge e are said to be incident with each other, as are v and e. If two distinct edges f and g are incident with a common vertex, then f and g are said to be adjacent edges.

Definition 2.4

An edge having the same vertex as both its end vertices is called a self loop orloop.

Definition 2.5

If there are more than one edge between the same given pair of vertices, these edges are called parallel edges.

STRONG EDGE - GRACEFUL LABELING OF GRAPHS

Introduction

Lo [17] introduced the notion of edge — graceful graphs. The necessary condition for a graph (p, q) to be edge-graceful is q(q+l) = 0 or (mod p).

With this condition, even cycles and paths of even length are not edge - graceful. But whether trees of odd order are edge - graceful is still open. On these lines, we define a new type of labeling called strong edge - graceful labeling by relaxing its range in which almost all even order trees are found to be strong edge - graceful.

So, in this chapter, we have introduced the strong edge - graceful labeling of graphs and established the strong edge - gracefulness of fan, twig, path, cycle, star, crown, spider and Pm, ΘnK1.

Definition 3.1.1

A graph G with q edges and p vertices is said to have an edge - graceful labeling (EGL) if there exists a bijectionf from the edge set to the set {1, 2, ..., q} so that the induced mapping f + from the vertex set to the set {0, 1, p-1} given by f + (x)

$=\Sigma\{f(xy)|\ xy\varepsilon\ \Sigma(G)\}$ (mod p) is a bijection. A graph G is edge - graceful graph (EGG)if it admits a edge - graceful labeling.

Definition 3.1.2

A (p, 4) graph G is said to have strong edge — graceful labeling (SEGL)if there exists an injection f from the edge set to 2 0 13 so that the 2 inducedmapping f+ from the vertex set to {0, 1, ..., 2p — 1} defined by (x) =I{f (xy)lxy E E(G)} (mod 2p) are distinct. A graph G is said to be strong edge - graceful graph(SEGG) if it admits a strong edge - graceful labeling.

For the clear understanding of the above definitions we illustrate the following examples.

3.1. **Fan graph***Fn*,

Definition 3.2.1

Let Pn, denote the path on n vertices. Then the join of K1 with Pn, is defined as fan and is denoted by Fn (i.e) Fn= K1 + Pn.

Theorem 3.2.2

The fan F4n_2 (n 2) is strong edge - graceful for all n.

Proof

Let {v, v1, v2, v3, ..., v,} be the vertices of F, and {vvi, : 1< i < t} {ei=(vi,vi+1):1<i<t- 1} be the edges of F, which are denoted as in Fig. 3.3.

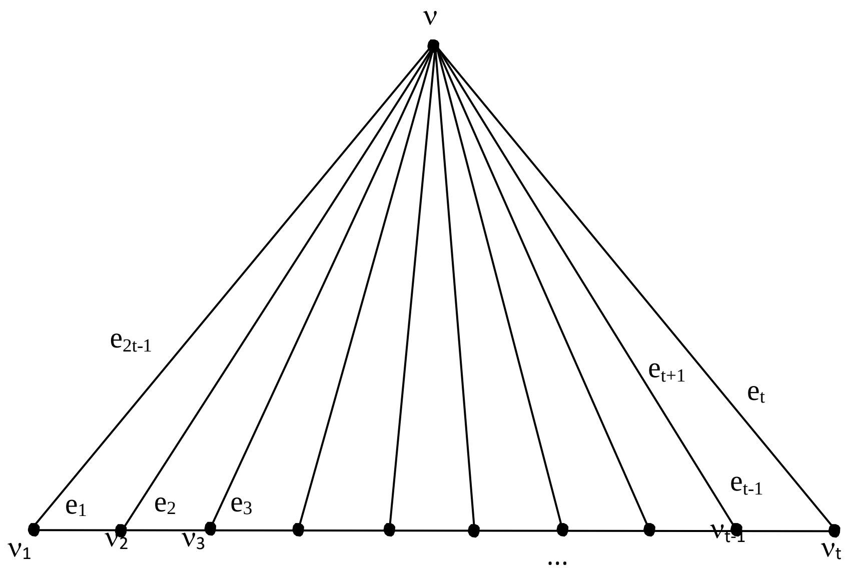
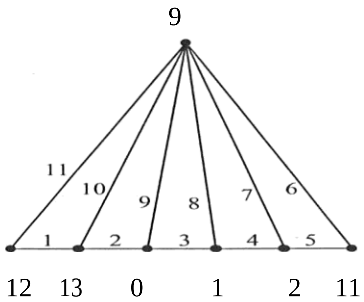


Fig. 3.3: Ft with ordinary labelling

Take t = 4n — 2.
We first label the edges of F4n_2 as follows:
f (ei) =i 1< i < 8n - 5
Then the induced vertex labels are
f+ (vi) = 2(4n - 2) + i - 1 1 < i< 2
f+ (v3)=0
f+ (v;) =i-3 4 < i 4n — 3
f+ (v4n-2) = 8n-5 f+(v) = 4n + 1

Clearly, the vertex labels are all distinct. Hence, the fan F4n-2 (n> 2) is strongedge -graceful for all n.
The SEGL of F6, Fl 0 are illustrated in Fig. 3.4 and Fig. 3.5 respectively.



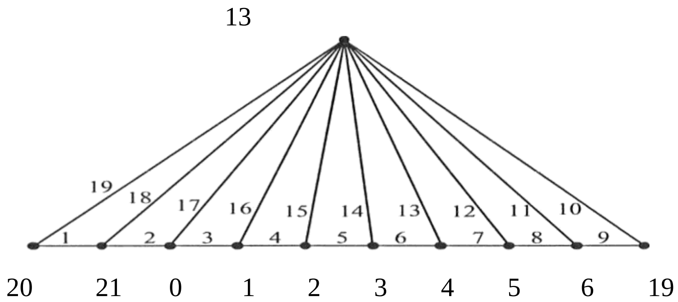


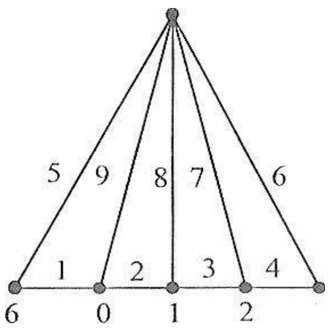
Fig. 3.4: F6 with SEGL **Fig. 3.5: Flo with SEGL**

The fan F4n+1 (n > 1) is strong edge - graceful for all n.

Proof

Let {v, v1, v2, v3, ...,vt} be the vertices of Ft and {vv1 : 1 < i < t} □ □ □ {e = (vi vi+1):1 < i < t-1} be the edges of Ft which are denoted as in Fig. 3.3, of Theorem

Take t = 4n + 1.
We first label the edges of F4n+1 as follows:
( ) {
Then the induced vertex labels are:
f+ (v_1) = 4n+2
f+ (v_i) = n-2 2 < i < 4 n
f+ (V_{4n+1}) = 8n+2 ; f+ (□) = 6n+5
Clearly, the vertex labels are all distinct. Hence, the fan F4n+1 (n > 1) is strong edge - graceful for all n.
The SEGL of F5, F9 are illustrated in Fig. 3.6 and Fig. 3.7 respectively.



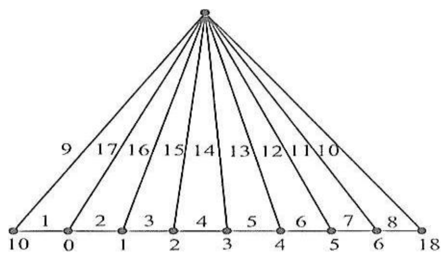


Fig. 3.6: F5 with SEGL Fig. 3.7: F9 with SEGL

EDGE GRACEFUL LABELING OF SOMETREES

4.1. Introduction

In Chapter III, we introduced strong edge - graceful labeling andestablished strong edge - graceful labeling of certain families of graphs. In this chapter, wediscuss edge - graceful labeling of some trees suchas

- ⟨K1n, :K1m⟩
- N⟨K1,n:K1,m⟩
- Yn•
- *FC* $(1^m, Ki,n)$
- *mGn*˙

4.2. Edge — gracefullabeling Definition

4.2.1 [17]

A graph *G(V, E)* is said to be edge - graceful if there exists a bijection*f* from *E* to {1, 2, ..., *q}* such that the induced mapping f^+ from *V* to {0, 1, 2, ..., p-1} given by *(x) = (Σf* (xy)) (modp) taken over all edges *xy*is abijection.

Lo [17] found a necessary condition for a graph with *p* vertices and *q* edges tobe edge - graceful as *q(q + 1)* = 0 or (mod*p).*

Definition 4.2.2

The graph (K1n, : K*1 m*) is obtained by joining the center *u* of the star K1, *n* andthe center v of another star *K1, m* to a new vertex w. The number of vertices is *n + m +* 3and the number of edges is*n + m* +2.

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Theorem4.2.3

The graph (IChn: K, m) is edge — graceful for all *n, in* even and *n in.*

Proof

Let{w,v,v',v1,v2,...,vn,vn+1,...,vn+m}betheverticesof*⟨K1n:K1,m⟩* and

{el, e2, en+m} ∪ *{e,* e'} be the edges of ⟨K1,n : K1,m⟩ which are denoted as in Fig. 4.1.

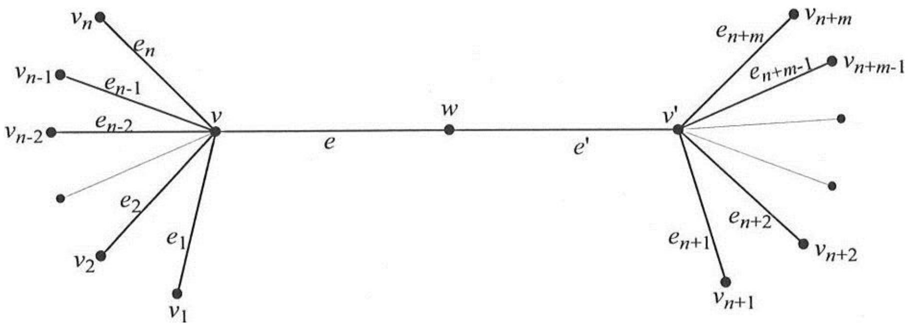

Fig. 4.1: *⟨K1,n*: K1,m⟩ with ordinarylabeling

We first label the edges asfollows:

ConsidertheDiophantineequationx1+x2=*p*.Thesolutionsareofthe form *(t, p — t)* where . There will be pair ofsolutions.
We first define *f (e) = q ; f (e')* = 1.

From other pairs, we label the edges of K1,nand K1,m by the coordinatesofthepairinanyordersothatadjacentedgesreceivethecoordinatesofthepairs. Then the induced vertex labelsare:

f^+ *(w)=0* ; f^+ *(v') =1* ; f^+ (v)=*q*

Now, the pendant vertices will have labels of the edges with which they are incident and they are distinct. Hence, the graph K1,n : K1,m is edge graceful for all n, m even and n #m.

CONCLUSION

In this paper we studied the graceful labelling the future extension work of this project will be conversion of the mathematical formulation of super graceful labelling for graphs into computer coding using any mathematical software this will help to obtain the super graceful labelling for more Complicated trees than caterpillars.
Graceful labelling have been studied for over decades and the topic continues to be a fascinating one in the world of graph theory and discrete mathematics an abcendance of published papers and results exist, yet various unsolved problems and unproven conjectures continue to allow for the en more research will the hopes that new results will be obtained.

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